Text of Intelligent Control

M. Yamakita

0.1 Sampled data system and its mathematical representation

0.1.1 Discretization of continuous linear time invariant (LTI) systems

We consider a system represented by a state space model :

$$\begin{aligned}\dot{x} &= Ax + Bu,\ x(0) = x_0 \\ y &= Cx + Du\end{aligned}$$

We assume that a zero order holder is inserted in front of the system and the output of the holder is denoted $u(kT)$, which is a input for the plant. The output of the plant is sampled at time instance kT and it is denoted $y(kT)$. If we consider $u(kT)$ and $y(kT)$ are input and output of the system, the system can be considered as a discrete time system which convert the input sequence $\{u(kT)\}$ into $\{y(kT)\}$. The input-output relationship of the system can be easily calculated as follows. Since the input for the continuous system is held constant during the sampling interval:

$$u(t) = u(kT) \quad (kT \leq t < (k+1)T) \tag{1}$$

and the solution of the differential equation (1) is given by

$$\begin{aligned}x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \qquad (2)\\ y(t) &= Cx(t) + Du(t), \qquad (3)\end{aligned}$$

$y(kT)$ is represented by

$$\begin{aligned}x(kT) &= e^{AkT}x_0 + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau \qquad (4)\\ y(kT) &= Cx(kT) + Du(kT). \qquad (5)\end{aligned}$$

Next, let consider the output at time $t = (k+1)T$. The output at the instance is similarly obtained as

$$\begin{aligned}x((k+1)T) &= e^{A(k+1)T}x_0 + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau \qquad (6)\\ y((k+1)T) &= Cx((k+1)T) + Du((k+1)T). \qquad (7)\end{aligned}$$

From this equations, we can easily obtain a recursive representation of $x(kT)$ as follows. Since

$$\begin{aligned}x((k+1)T) &= e^{AT}e^{AkT}x_0 + e^{AT}\int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau \\ &+ \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau \qquad (8)\end{aligned}$$

and $u(t)$ is $u(kT)$ over$[kT, (k+1)T)$, we have

$$x((k+1)T) = e^{AT}x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau. \tag{9}$$

Moreover, the second term in the above equation can be modified as

$$\int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau = \int_0^T e^{A\tau}d\tau Bu(kT). \tag{10}$$

If we define the following constant matrices;

$$\Phi := e^{AT} \quad \Gamma := \int_0^T e^{A\tau}d\tau B, \tag{11}$$

$y(kT)$ can be represented using$(x(kT), u(kT))$ as

$$
\begin{aligned}
x((k+1)T) &= \Phi x(kT) \ + \ \Gamma u(kT) \qquad (12)\\
y(kT) &= Cx(kT) \ + \ Du(kT). \qquad (13)
\end{aligned}
$$

This difference equation is a discrete time state space representation of the continuous system. Please notice that once the sampling interval T is given, the matrices, (Φ, Γ), are calculated as constant matrices. As in the preceding procedure, determination of the discrete time state space representation from the continuous one is called discretization of a system. If we should know the state value during the interval, it can be determined by

$$
\begin{aligned}
x(kT+\tau) &= e^{A\tau}x(kT) \ + \ \int_0^{\tau} e^{Ar}drBu(kT), \ \ .(0 \ \leq \ \tau \ < \ T) \qquad (14)\\
y(kT+\tau) &= Cx(kT+\tau) \ + \ Du(kT) \qquad (15)
\end{aligned}
$$

z transformation and pulse transfer function

In the previous section we have learned how to obtain a discrete time state space representation form the continuous one. Of course, we can use the representation for controller design, however, sometimes input-output property is required. In a continuous time case, input-output relationship is represented by a transfer function using Laplace transformation. In the discrete time case, the Laplace transformation is replaced by z-transformation. Before we mention the input-output relationship of the system, how to solve difference equations using z-transformation, which is similar to how to solve differential equations using Laplace transformation. The z-transformation is define as follow.

[Definition] z transformation of a sequence $\{f(kT)\}$

$$\mathcal{Z}\{f(kT)\} \ := \ F(z) \ = \ \sum_{k=0}^{\infty} f(kT)z^{-k} \qquad (16)$$

Properties of z transformation can be listed as follows;

1. (Linearity)

$$\mathcal{Z}\{c_1 f_1 + c_2 f_2\} \ = \ c_1 \mathcal{Z}\{f_1\} \ + \ c_2 \mathcal{Z}\{f_2\}$$

2. (Shift)

$$
\begin{aligned}
\mathcal{Z}\{f((k+n)T)\} &= z^n(\mathcal{Z}\{f\} \ - \ \sum_{l=0}^{n-1} f(lT)z^{-l})\\
\mathcal{Z}\{f((k-n)T)\} &= z^{-n}\mathcal{Z}\{f\}
\end{aligned}
$$

3. (Convolution)

$$\mathcal{Z}\{f * g\} \ = \ \mathcal{Z}\{f\} \cdot \mathcal{Z}\{g\}$$

   ただし、

$$f * g \ := \ \sum_{l=0}^{k} f((k-l)T)g(lT)$$

4. (Inverse formula)

$$f(kT) \ = \ \frac{1}{2\pi j} \oint F(z)z^{k-1}dz$$

5. (Initial value theorem)

$$f(0) \ = \ \lim_{z \to \infty} F(z)$$

6. (Final value theorem)

$$f(\infty) \;=\; \lim_{z\to 1}(1-z^{-1})F(z)$$

7. (Parseval theorem)

$$\sum_{k=0}^{\infty} f^2(kT) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(e^{j\theta})|^2 d\theta$$

As a transfer function is defined as a fraction of a Laplace transformation of output signal and a Laplace transformation of input signal, a pulse transfer function of a system is defined as a fraction of a z-transformation of output sequence and z-transformation of input sequence. The pulse transfer function can be also obtained using a discrete time state space representation. That is, z-transforming (12) as

$$zX(z) - zx(0) \;=\; \Phi X(z) \;+\; \Gamma U(z), \tag{17}$$

if we assume $x(0) = 0$, we have

$$X(z) \;=\; (zI - \Phi)^{-1}\Gamma U(z).$$

Therefore, the pulse transfer function is determined as

$$H(z) \;=\; C(zI - \Phi)^{-1}\Gamma \;+\; D. \tag{18}$$

[Example] Discretize the following LTI system and obtain its pulse transfer function.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [1\;\;0]x \end{aligned}$$

Since the matrix A has a special property, (Φ, Γ) can be easily determined as follows;

$$\begin{aligned} \Phi &= I + A + 0 + \ldots = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \\ \Gamma &= \int_0^T e^{A\tau} B \;=\; \int_0^T I + A\tau d\tau B \;=\; \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix}. \end{aligned}$$

From the definition of a pulse transfer function, the corresponding pulse transfer function is calculated as

$$H(z) \;=\; [1\;\;0]\left(zI - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix} \;=\; \frac{T^2(z+1)}{2(z-1)^2}.$$

Pulse transfer function and its frequency property

In the previous section, pulse transfer function is introduced in order to characterize input-output relationship of discrete time systems. Since the pulse transfer function is not defined based on continuous time t, it has no relationship to the sampling interval T. If we design a controller for continuous time plant using a sampler and holder, frequency property of the controlled system is very important, however, the frequency property is not linked directly so far. In this section, how to extract the frequency property from the pulse transfer function is discussed.

As is well known that if we inject a input:

$$u(t) \;=\; sin(\omega t) \tag{19}$$

into a stable continuous LTI system, the output of the system can be represented by れる。

$$y(t) \;=\; |H(j\omega)| sin(\omega t + ArgH(j\omega)) \tag{20}$$

after a transient, where $H(s)$ is the transfer function of the system. In this case $H(j\omega)$ is referred to a frequency transfer function. This equation means that the frequency transfer function is obtained by substituting $j\omega$ into s if the plant is stable. $H(j\omega)$ contains information how the magnitude and phase of the injected sinusoidal signal whose natural frequency is ω are influenced by the system.

Next, we consider how to obtain the frequency transfer function in a discrete time case. Let consider to inject a input signal:

$$u(kT) = sin\omega kT \tag{21}$$

into a system whose pulse transfer function is $H(z)$. Using z-transformation, $Y(z)$ is obtained as follows:

$$\begin{aligned} U(z) &= \mathcal{Z}\{u(kT)\} = \frac{zsin\omega T}{z^2 - 2zcos\omega T + 1} && (22) \\ Y(z) &= H(z)U(z). && (23) \end{aligned}$$

Using the inverse formula, the inverse transform of $Y(z)$ is calculated by

$$\begin{aligned} y(kT) &= \mathcal{Z}^{-1}\{H(z)\frac{zsin\omega T}{z^2 - 2zcos\omega T + 1}\} \\ &= \frac{1}{2\pi j}\oint H(z)\frac{z^k sin\omega T}{z^2 - 2zcos\omega T + 1}dz. && (24) \end{aligned}$$

If $H(z)$ does not have any pole on a unit circle, the output is calculated as

$$y(kT) = \frac{H(e^{j\omega T})e^{j\omega kT}}{2} - \frac{H(e^{-j\omega T})e^{-j\omega kT}}{2} + \sum_{|\alpha|<1} R(\alpha)\alpha^k. \tag{25}$$

Moreover, if k is large enough, a third term in the above equation vanishes, we have

$$y(kT) = |H(e^{j\omega T})|sin(\omega kT + ArgH(e^{j\omega T})). \tag{26}$$

From this equation, we can see that the frequency transfer function of the pulse transfer function can be obtained by substituting $e^{j\omega T}$ into z. Moreover, if we substitute $\omega + \omega_s$ in stead of ω, where

$$\omega_s = \frac{2\pi}{T}, \tag{27}$$

the value does not change since

$$H(e^{j(\omega+\omega_s)T}) = H(e^{j\omega T}). \tag{28}$$

Therefore, the frequency property of $H(z)$ is a periodic function whose period is ω_s. This property is due to that if we inject a signal:

$$u(t) = sin(\omega + \omega_s)kT \tag{29}$$

into a sample and holder, the signal can not be distinguished from $u(t) = sin\omega kT$ by sampling operation. This kind of phenomena by sampling is referred to aliasing which has strong relationship to well known sampling theorem, however, it is not discussed here.

Chapter 1

State Space Representation, Shift Operators, and DARMA Model

We are now familiar to state space representation and its pulse transfer function. The pulse transfer function represents input-output relationship without the intermediate variables, states, in z domain. Now we consider a time domain representation of the input-output relationship of the systems.

1.1 Forward and Backward Shift Operators

Forward and backward operators, q and q^{-1} are defined as follows:

$$qy(k) = y(k+1) \quad (k \geq 0) \tag{1.1}$$

$$q^{-1}y(k) = \begin{cases} y(k-1) & (k \geq 1) \\ 0 & (k = 0) \end{cases} \tag{1.2}$$

and consequently,

$$q^{i}y(k) = y(k+i) \quad (k \geq 0) \tag{1.3}$$

$$q^{-i}y(k) = \begin{cases} y(k-i) & (k \geq i) \\ 0 & (0 \leq k < i) \end{cases} \tag{1.4}$$

Please note that the shift operator q seems to be very similar to z transformation, however, it has very different property as they 'ignore' initial value of the signal though it is always taken account into in the z transformation.(That is, if we ignore the initial value, q is equivalent to z in a form.)

In general input-output relationship of time invariant linear system can be represented as

$$P(q)z(k) = Q(q)u(k), \quad k \geq 0 \tag{1.5}$$

$$y(k) = R(q)z(k), \quad z(k) \in R^{l},\ u(k) \in R^{m}\ \ y(k) \in R^{p} \tag{1.6}$$

using the forward shift operator q where $z(k)$ is a 'partial' state. To ensure existence and uniqueness of the solution to (1.6), we require $P(q)$ to be square and nonsingular for almost q.

The difference operator representation includes the state space representation as a special case when

$$P(q) = qI - A \tag{1.7}$$

$$Q(q) = B \tag{1.8}$$

$$R(q) = C \tag{1.9}$$

$$z(k) = x(k) \tag{1.10}$$

1.1.1 Left Difference Operator Representation and Observer Form

As a special case of the difference operator representation, left difference operator representations are given by

$$\begin{aligned} D_L(q)z(k) &= N_L(q)u(k), \quad k \geq 0 & (1.11)\\ y(k) &= z(k), & (1.12)\end{aligned}$$

or equivalently,

$$D_L(q)y(k) = N_L(q)u(k). \tag{1.13}$$

If we consider a single-input single-output (scaler) system whose pulse transfer function $H(z)$:

$$H(z) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n}{a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}, \tag{1.14}$$

then ignoring the initial response we have

$$Y(z) = H(z)U(z). \tag{1.15}$$

Therefore, we have

$$(a_0 z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n)Y(z) = (b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n)U(z), \tag{1.16}$$

and since now z can be considered as a forward shift operator, it can be represented as

$$(a_0 q^n + a_1 q^{n-1} + \cdots + a_{n-1} q + a_n)y(k) = (b_1 q^{n-1} + b_2 q^{n-2} + \cdots + b_n)u(k) \tag{1.17}$$

in the time (index) domain k. From the pulse transfer function we have derived a particular left difference operator representation of the system where

$$\begin{aligned} D_L(q) &= a_0 q^n + a_1 q^{n-1} + \cdots + a_{n-1} q + a_n & (1.18)\\ Q_L(q) &= b_1 q^{n-1} + b_2 q^{n-2} + \cdots + b_n. & (1.19)\end{aligned}$$

Once we have a left difference operator representation of the system, a special state space representation can be easily obtained as follows. In (1.17) operating shift operator and shifting the index, we have

$$y(k) = \frac{a_1}{a_0} y(k-1) + \cdots + \frac{a_n}{a_0} y(k-n) + \frac{b_1}{a_0} u(k-1) + \cdots + \frac{b_n}{a_0} u(k-n), \tag{1.20}$$

where we assume $a_0 \neq 0$. Introducing new variables so that the right-hand side of the definition of the new variable just contains the term whose index is just smaller than 1, i.e.

$$\begin{aligned} y(k) &= -\frac{a_1}{a_0} y(k-1) - \cdots - \frac{a_n}{a_0} y(k-n) + \frac{b_1}{a_0} u(k-1) + \cdots + \frac{b_n}{a_n} u(k-n) & (1.21)\\ &= \frac{a_1}{a_0} y(k-1) + r_1(k-1) + \frac{b_1}{a_0} u(k-1) & (1.22)\\ r_1(k) &= -\frac{a_2}{a_0} y(k-1) + r_2(k-1) + \frac{b_2}{a_0} u(k-1) & (1.23)\\ &\cdots & (1.24)\\ r_{n-1}(k) &= -\frac{a_n}{a_0} y(k-1) + \frac{b_n}{a_0} u(k-1) & (1.25)\end{aligned}$$

Now we define a state vector as

$$x(k) = [y(k), r_1(k), \cdots, r_{n-1}(k)]^T \tag{1.26}$$

leading to the following state state representation:

$$x(k+1) = \begin{bmatrix} -\frac{a_1}{a_0} & 1 & . & . & 0 \\ -\frac{a_2}{a_0} & 0 & 1 & . & 0 \\ & & \cdots & & 0 \\ -\frac{a_{n-1}}{a_0} & & \cdots & & 1 \\ -\frac{a_n}{a_0} & 0 & . & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} \frac{b_1}{a_0} \\ . \\ \\ . \\ \frac{b_n}{a_0} \end{bmatrix} u(k) := A_o x(k) + b_o u(k) \tag{1.27}$$

$$y(k) = [1, 0, \cdots, 0] x(k) := c_o x(k) \tag{1.28}$$

which is called observer form or observable canonical form. The above representation is called observer form since an observer can be easily designed based on the observer form. Let $\hat{x}$ be the estimated state of the system and the observer dynamics is given by

$$\hat{x}(k+1) = A_o \hat{x}(k) + b_o u(k) - k_o(\hat{y}(k) - y(k)) \tag{1.29}$$

where

$$k_o = [k_0, k_1, \cdots, k_n]. \tag{1.30}$$

Then the error system of the state is geven by

$$\tilde{x}(k+1) = (A_o - b_o k_o)\tilde{x}(k), \quad \tilde{x} := \hat{x}(k) - x(k). \tag{1.31}$$

Since we have

$$A_o - K_o c_o = \begin{bmatrix} -k_1 - \frac{a_1}{a_0} & 1 & . & . & 0 \\ -k_2 - \frac{a_2}{a_0} & 0 & 1 & . & 0 \\ & & \cdots & & 0 \\ -k_{n-1} - \frac{a_{n-1}}{a_0} & & \cdots & & 1 \\ -k_n - \frac{a_n}{a_0} & 0 & . & 0 & 0 \end{bmatrix}, \tag{1.32}$$

and $det(zI - (A_o - K_o c_o)) = z^n + (k_1 + \frac{a_1}{a_0})z + \cdots + k_n + \frac{a_n}{a_0}$, $\{k_i\}$ is easily determined if the desired pole allocation of the observer is assigned. Please note here that once the observer form is obtained, the backword operations in the preceding discussion gives the representation of (1.20). In the preceding discussion, the observer form was derived from a transfer function, however, it will be dangerous for the design, especially for servo controller design, since uncontrollable modes may disappear in the model. For the controller design purpose, the state space model should be transformed into observer form from the original state space model but from the transfer function. See the details in an example below.

1.1.2 DARMA Model

From the above derivation, it has been seen that if the expression as in (1.20) is given, the other expressions can easily obtained. The expression itself, however, has very natural meaning how the current output is influenced by the past output and the input. The model like (1.20) is called DARMA model. The name DARMA stands for Deterministic Auto Regressive Moving Average where 'deterministic' means that there is no stochastic term in the model. ARMA is a model of a stochastic system and is also combination of AR and MA. If the ARMA model has no input $u(k)$ but stochastic signal $\omega(k)$, i.e.

$$y(k) = \frac{a_1}{a_0} y(k-1) + \cdots + \frac{a_n}{a_0} y(k-n) + c_1 \omega(k-1) + \cdots \tag{1.33}$$

it is called AutoRegressive model. If ARMA model has only stochastic term, i.e.

$$y(k) = c_1 \omega(k-1) + \cdots \tag{1.34}$$

it is called a moving average model. If the right-hand side of the equation contains all terms, i.e.

$$y(k) = \frac{a_1}{a_0} y(k-1) + \cdots + \frac{a_n}{a_0} y(k-n) + \frac{b_1}{a_0} u(k-1) + \cdots + \frac{b_n}{a_n} u(k-n) + c_1 \omega(k-1) \cdots, \tag{1.35}$$

it is called ARMAX model where 'X' stands for auXiliary input $u(k)$ and ARMAX model is just a DARMA model with stochastic signal $\omega(k)$.

In general DARMA model is defined by

$$A_0 y(k) = -\sum_{j=1}^{n_1} A_j y(k-j) + \sum_{j=0}^{m_1} B_j u(k-j-d), \quad k \geq 0 \tag{1.36}$$

where $y(k) \in R^p, u(k) \in R^m$, A_0 is square and nonsingular, A_j, B_j are appropriate matrices, and d stands for time delay.

Using the backword shift operator q^{-1}, (1.36) can be expressed as

$$A(q^{-1})y(k) = B(q^{-1})u(k) \tag{1.37}$$

where

$$A(q^{-1}) = A_0 + A_1 q^{-1} + \cdots + A_{n_1} q^{-n_1} \tag{1.38}$$

$$B(q^{-1}) = (B_0 + B_1 q^{-1} + \cdots + B_{m_1} q^{-m_1}) q^{-d} \tag{1.39}$$

As shown in the previous section, this model is just one of a left difference operator representation.

Moreover, (1.36) can be expressed as

$$y(k) = \phi^T(k)\theta_0, \tag{1.40}$$

where $\theta_0^T \in R^{p\times\nu}$ is a parameter matrix (low vector for scalar case) which contains elements of A_j and B_j, $\phi \in R^{\nu\times n_1+m_1}$ is called a regression matrix (regression vector for scalar case) which contains $\{y(k-1), \cdots, y(k-n_1), u(k-d), \cdots, u(k-m_1-d)\}$. (Note that in the scalar case, $\theta_0^T\phi = \phi^T\theta_0$.)

[Example]

Let consider a system ,which is an one-step delay element with constant disturbance, given by the following difference equations:

$$\begin{aligned} \eta(k+1) &= \eta(k) \\ z(k+1) &= u(k) \\ y(k) &= z(k) + \eta(k) \end{aligned}$$

The problem here is to obtain a DARMA model for the system. Introducing a state vector $x(k)$ as

$$x(k) = [\eta(k), z(k)]^T,$$

it can be represented as

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= [1, 1]x(k). \end{aligned}$$

Using the PBH(Popov-Belevitch-Hautus) tests, i.e.,

$$rank C = rank[b, Ab] = 1 \tag{1.41}$$

$$rank N^T = rank[c^T, A^T c^T] = 2, \tag{1.42}$$

it can be seen that the system is not completely controllable but completely observable. Then the all state will give influences to the output.

First we will consider a 'wrong' derivation. Let consider a pulse transfer function $H(z)$. The transfer pulse function is given by

$$H(z) = c(zI-A)^{-1}b = [1 \;\; 1]\frac{\begin{bmatrix} z & 0 \\ 0 & z-1 \end{bmatrix}}{z(z-1)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{z-1}{z(z-1)} = \frac{1}{z}$$

As shown in the above equation, the uncontrollable mode $z = 1$ is cancelled out. If we use this transfer function to obtain the DARMA model, we have

$$y(k) = u(k-1).$$

Since this model does not include the information of the disturbance $\eta(k)$, controllers based on this model may cause an offset in the output if $\eta(0) \neq 0$.

The 'right' procedure is given as follows. Using a dual procedure given in Furuta, we can employ a coordinate transformation matrix T as

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},$$

and the equivalent system is given by

$$\begin{aligned} \bar{x}(k+1) &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \bar{x}(k) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(k) \\ y(k) &= [0 \quad 1]\bar{x}. \end{aligned}$$

Furthermore, reversing the order of the elements in $\bar{x}$ gives

$$\begin{aligned} \bar{x}(k+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \bar{x}(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k) \\ y(k) &= [1 \quad 0]\bar{x}. \end{aligned}$$

From this observer form we directly obtain the following DARMA model:

$$y(k) = y(k-1) + u(k-1) - u(k-2). \tag{1.43}$$

Moreover we have a form using a regressor vector as

$$y(k) = [y(k-1) \;\; u(k-1) \;\; u(k-2)] \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = [1 \;\; 1 \;\; -1] \begin{bmatrix} y(k-1) \\ u(k-1) \\ u(k-2) \end{bmatrix}$$

Of course in this case, since we know that the system is completely observable, we can derive the above DARMA model from the transfer function without the cancellation.

Chapter 2

Identification of Linear Discrete Time Systems

2.1 Projection Algorithm

Suppose that a **scalar** discrete time system is represented by the following equation:

$$y(k) = \phi^T(k-1)\theta_0 \tag{2.1}$$

where $\phi^T(k-1)$ is a regression vector (matrix) and θ_0 is a true parameter vector as :

$$\begin{aligned} \phi^T(k-1) &:= [y(k-1), \cdots, y(k-n), u(k-1-d), \cdots, u(k-l-d)]\,,\ d=1 && (2.2)\\ \theta_0^T &:= [\theta_1, \cdots, \theta_{n+l}]\,. && (2.3)\end{aligned}$$

The projection algorithm for this system is given by the following iteration.

$$\hat{\theta}(k) = \hat{\theta}(k-1) \ + \ \frac{a\phi(k-1)}{c+\phi^T(k-1)\phi(k-1)}\left[y(k) - \phi^T(k-1)\hat{\theta}(k-1)\right] \tag{2.4}$$

where a, c are scalar constants satisfying

$$\begin{aligned} c &> 0\\ 0 &< a < 2. \end{aligned}$$

Now we define the parameter error vector and error signal as

$$\begin{aligned} \tilde{\theta}(k) &:= \hat{\theta}(k) - \theta_0 && (2.5)\\ e(k) &:= y(k) - \phi^T(k-1)\hat{\theta}(k-1)\\ &= -\phi^T(k-1)\tilde{\theta}(k-1) && (2.6)\end{aligned}$$

then we have the following properties about the projection algorithm.
[Lemma]

1. $$\|\hat{\theta}(k) - \theta_0\| \ \le \ \|\hat{\theta}(0) - \theta_0\|,\ k \ge 1 \tag{2.7}$$

2. $$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{e^2(k)}{c+\phi^T(k-1)\phi(k-1)} \ < \ \infty \tag{2.8}$$

   and

(a)

$$\lim_{k\to\infty} \frac{e(k)}{[c+\phi^T(k-1)\phi(k-1)]^{1/2}} = 0 \tag{2.9}$$

(b)

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{\phi^T(k-1)\phi(k-1)e^2(k)}{c+\phi^T(k-1)\phi(k-1)} \ < \ \infty \tag{2.10}$$

(c)

$$\lim_{N\to\infty} \sum_{k=1}^{N} \|\hat{\theta}(k) - \hat{\theta}(k-1)\|^2 \ < \ \infty \tag{2.11}$$

(d)

$$\lim_{N\to\infty} \sum_{k=M}^{N} \|\hat{\theta}(k) - \hat{\theta}(k-M)\|^2 \ < \ \infty, \ 1 \ \geq \ M \ < \ \infty \tag{2.12}$$

(e)

$$\lim_{k\to\infty} \|\hat{\theta}(k) - \hat{\theta}(k-M)\|^2 = 0 \tag{2.13}$$

[Note]
The last equation does not mean that $\hat{\theta}(k)$ converges to a parameter vector. Consider a sequence $\{S_k\}$ given by

$$S_k \ := \ \sum_{i=1}^{k} \frac{1}{i}.$$

We can check easily that $\lim_{k\to\infty} |S_k - S_{k-M}| = 0$ but $\lim_{k\to\infty} S_k = \infty$.
[Proof]

1. Subtracting θ_0 from both sides of (2.4), we have

$$\tilde{\theta}(k-1) = \tilde{\theta}(k) - \frac{a\phi(k-1)}{c+\phi^T(k-1)\phi(k-1)}\phi^T(k-1)\tilde{\theta}(k-1). \tag{2.14}$$

Using (2.6), it follows that

$$\|\tilde{\theta}(k)\|^2 - \|\tilde{\theta}(k-1)\|^2 = a\left[-2 + \frac{a\phi^T(k-1)\phi(k-1)}{c+\phi^T(k-1)\phi(k-1)}\right]\frac{e^2(k)}{c+\phi^T(k-1)\phi(k-1)}. \tag{2.15}$$

On the other hand, from the condition $0 \ < \ a \ < \ 2$ and

$$a\left[-2 + \frac{a\phi^T(k-1)\phi(k-1)}{c+\phi^T(k-1)\phi(k-1)}\right] \ < \ a(a-2) \ < \ 0, \tag{2.16}$$

(2.7) is derived.

2. Adding both sides of (2.15) from j=1 to k, we have

$$\|\tilde{\theta}(k)\|^2 = \|\tilde{\theta}(0)\|^2 + \sum_{j=1}^{k} a\left[-2 + \frac{a\phi^T(k-1)\phi(k-1)}{c+\phi^T(k-1)\phi(k-1)}\right]\frac{e^2(k)}{c+\phi^T(k-1)\phi(k-1)}. \tag{2.17}$$

On the other hand, from the condition of a, the following inequality is satisfied.

$$\|\tilde{\theta}(k)\|^2 \leq \|\tilde{\theta}(0)\|^2 + a(a-2)\sum_{j=1}^{k} \frac{e^2(k)}{c+\phi^T(k-1)\phi(k-1)}. \tag{2.18}$$

Therefore,

$$a(2-a)\sum_{j=1}^{k}\frac{e^2(k)}{c+\phi^T(k-1)\phi(k-1)} \ \le\ \|\tilde{\theta}(0)\|^2-\|\tilde{\theta}(k)\|^2, \tag{2.19}$$

and taking $k\to\infty$ leads to (2.8).

(a) (2.9) is obvious from (2.8).

(b) From

$$\frac{e^2(k)}{c+\phi^T(k-1)\phi(k-1)} = \frac{(c+\phi^T(k-1)\phi(k-1)e^2(k)}{(c+\phi^T(k-1)\phi(k-1))^2} \ \ge\ \frac{\phi^T(k-1)\phi(k-1))e^2(k)}{(c+\phi^T(k-1)\phi(k-1))^2}, \tag{2.20}$$

the relation is obvious.

(c) (2.4) and (2.10) lead to the end.

(d) From

$$\|\hat{\theta}(k)-\theta(k-M)\|^2 = \|\hat{\theta}(k)-\theta(k-1)+\theta(k-1)-\cdots+\theta(k-M+1)-\theta(k-M)\|^2$$

and

$$\|a+b\|^2 \ \le\ 2(\|a\|^2+\|b\|^2),$$

we have

$$\|\hat{\theta}(k)-\theta(k-M)\|^2 \ \le\ C(M)(\|\hat{\theta}(k)-\theta(k-1)\|^2+\|\theta(k-1)-\cdots+\|\theta(k-M+1)-\theta(k-M)\|^2),\ C(M)<\infty,$$

and (2.12) is derived.

(e) It is obvious from(2.12).

2.2 Least Square Method

If we use the projection algorithm when noises are present at system equation and in the measurements, the parameter vector does not converge to a constant value and are changing due to the noises. The least square method, however, will give a true value in that case because of the averaging property.

The algorithm is given by the following iteration:

$$\hat{\theta}(k) \ = \ \hat{\theta}(k)+\frac{P(k-2)\phi(k-1)}{1+\phi^T(k-1)P(k-2)\phi(k-1)}e(k-1) \tag{2.21}$$

$$P(k-1) \ = \ P(k-2)-\frac{P(k-2)\phi(k-1)\phi^T(k-1)P(k-2)}{1+\phi^T(k-1)P(k-2)\phi(k-1)}, \tag{2.22}$$

where $P(-1)$ is an arbitrary positive definite matrix P_0.

Before considering the above algorithm, we have the following lemma.

[Lemma] (Matrix Inversion Lemma)

Assume A is nonsingular and $A+BC$ is also nonsingular, the following equality is satisfied:

$$(A+BC)^{-1} \ = \ A^{-1} \ - \ A^{-1}B(I \ + \ CA^{-1}B)^{-1}CA^{-1} \tag{2.23}$$

[Proof] Multiplying $A+BC$ to both sides and observe that results become identity matrix.

If $P^{-1}(k)$ satisfies the following equation:

$$P^{-1}(k-1) = P^{-1}(k-2) \ + \ \phi(k-1)\phi^T(k-1)a(k-1), \tag{2.24}$$

the above lemma shows the following equation:

$$\begin{aligned} P(k-1) &= P(k-2) - \frac{P(k-2)\phi(k-1)\phi^T(k-1)P(k-2)a(k-1)}{1+\phi^T(k-1)P(k-2)\phi(k-1)a(k-1)} && (2.25)\\ P(k-1)\phi(k-1) &= \frac{P(k-2)\phi(k-1)}{1+\phi^T(k-1)P(k-2)\phi(k-1)a(k-1)}. && (2.26)\end{aligned}$$

The least square algorithm is derived considering the following function as the criterion function.

$$J_N = \frac{1}{2}\sum_{k=1}^{N}\|y(k)-\phi^T(k-1)\hat{\theta}(N)\|^2 + \frac{1}{2}(\hat{\theta}(N)-\hat{\theta}(0))^T P_0^{-1}(\hat{\theta}(N)-\hat{\theta}(0)) \tag{2.27}$$

Now we define Y_N and Φ_N using the observed output and injected input as

$$\begin{aligned} Y_N^T &:= [y(1),\cdots,y(N)] && (2.28)\\ \Phi_{N-1} &:= [\phi(0),\cdots,\phi(N-1)]^T. && (2.29)\end{aligned}$$

Using the above definitions, we have

$$\begin{aligned} \begin{bmatrix} e(1)\\ \cdot\\ e(N)\end{bmatrix} &= \begin{bmatrix} y(1)\\ \cdot\\ y(N)\end{bmatrix} - \begin{bmatrix} \phi^T(0)\\ \cdot\\ \phi^T(N-1)\end{bmatrix}\hat{\theta}(k-1)\\ &= Y_N - \Phi_{N-1}\theta. && (2.30)\end{aligned}$$

Using this relationship (2.27) can be expressed as

$$J_N = \frac{1}{2}(Y_N-\Phi_{N-1}\hat{\theta}(N))^T(Y_N-\Phi_{N-1}\hat{\theta}(N)) + \frac{1}{2}(\hat{\theta}(N)-\hat{\theta}(0))^T P^{-1}(0)(\hat{\theta}(N)-\hat{\theta}(0)). \tag{2.31}$$

Using the expression, the optimal estimated value $\hat{\theta}(N)$ is given taking the the partial derivative of the criterion function with the argument as

$$\frac{\partial J_N}{\partial\hat{\theta}(N)} = 0. \tag{2.32}$$

Therefore, we have

$$-\Phi_{N-1}^T(Y_N-\Phi_{N-1}\hat{\theta}(N)) + P^{-1}(0)(\hat{\theta}(N)-\hat{\theta}(0)) = 0. \tag{2.33}$$

Then $\hat{\theta}(N)$ is expressed by

$$\begin{aligned} \hat{\theta}(N) &= \left[\Phi_{N-1}^T\Phi_{N-1}+P^{-1}(0)\right]^{-1}\left(P_0^{-1}\hat{\theta}(0)+\Phi_{N-1}^T Y_N\right)\\ &= P(N-1)\left(P_0^{-1}\hat{\theta}(0)+\Phi_{N-1}^T Y_N\right), && (2.34)\end{aligned}$$

where

$$P^{-1}(N-1) = \Phi_{N-1}^T\Phi_{N-1} + P_0^{-1}. \tag{2.35}$$

The above equation can be also expressed as

$$\begin{aligned} P^{-1}(N-1) &= \Phi_{N-1}^T\Phi_{N-1} + P^{-1}(0)\\ &= \left[\ \phi(0)\ \ \cdots \mid \phi(N-1)\ \right]\begin{bmatrix} \phi^T(0)\\ \cdot\\ \hline \phi^T(N-1)\end{bmatrix} + P_0^{-1} && (2.36)\\ &= \left[\ \Phi_{N-2} \mid \phi(N-1)\ \right]\begin{bmatrix} \Phi_{N-2}^T\\ \hline \phi^T(N-1)\end{bmatrix} + P_0^{-1}\\ &= \Phi_{N-2}\Phi_{N-2}^T + P_0^{-1} + \phi(N-1)\phi^T(N-1)\\ &= P^{-1}(N-2) + \phi(N-1)\phi^T(N-1). && (2.37)\end{aligned}$$

On the other hand, $\hat{\theta}(N)$ satisfies the following equation:

$$\hat{\theta}(N) = P(N-1)(P_0^{-1}\hat{\theta}(0) + \Phi_{N-2}Y_{N-1} + \phi(N-1)y(N)), \tag{2.38}$$

and using (2.34), we have

$$\begin{aligned}
&= P(N-1)(P^{-1}(N-2)\hat{\theta}(N-1) + \phi(N-1)y(N)) \\
&= P(N-1)(P^{-1}(N-1) - \phi(N-1)\phi^T(N-1))\hat{\theta}(N-1) + P(N-1)\phi(N-1)y(N) \\
&= \hat{\theta}(N-1) + P(N-1)\phi(N-1)(y(N) - \phi^T(N-1)\hat{\theta}(N-1)).
\end{aligned} \tag{2.39}$$

Furthermore, using (2.26), we have

$$= \hat{\theta}(N-1) + \frac{P(N-2)\phi(N-1)}{1+\phi^T(N-1)P(N-2)\phi(N-1)}(y(N) - \phi^T(N-1)\hat{\theta}(N-1)), \tag{2.40}$$

and (2.21) is derived. (2.22) is obvious from the matrix inversion lemma.

The least square algorithm has the following properties.

[Lemma]

1. $$\|\hat{\theta}(k) - \theta_0\|^2 \ \leq \ \kappa_1\|\hat{\theta}(0) - \theta_0\|^2, \ k \ \geq \ 1 \tag{2.41}$$

   where

   $$\kappa_1 \ = \ Cond(P_0^{-1}) \ := \ \frac{\lambda_{max}(P_0^{-1})}{\lambda_{min}(P_0^{-1})}$$

2. $$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{e^2(k)}{1+\phi(k-1)P(k-2)\phi^T(k-1)} \ < \ \infty \tag{2.42}$$

   $$\lim_{k\to\infty}\|\hat{\theta}(k) - \hat{\theta}(k-M)\| = 0, \ for \ M \ < \ \infty \tag{2.43}$$

[Proof] Omitted.

Chapter 3

Linear Diophantine Equation and Adaptive Predictor

3.1 Linear Diophantine Equation

Suppose that a、b、c and d are polynomial of d. (In the analysis of the control system we consider d as $d = q^{-1}$.) The following equation in terms of x and y expressed by (3.1) is called Linear Diophantine Equation.

$$ax + by = c \tag{3.1}$$

A fundamental property about the Diophantine equation is given by the following theorem.
[Theorem]
Suppose that (a, b) is the largest common factor of a and b. The Diophantine equation (3.1) has a solution if and only if (a, b) is a common factor of c. We denote the condition as

$$(a, b)|c. \tag{3.2}$$

[Proof]
(Only if)
Assume that x_0 and y_0 are the solution of (3.1). If $g = (a, b)$, we have

$$a = ga_0,\ b = gb_0, \tag{3.3}$$

therefore the equation can be represented as

$$ax + by = g(a_0x + b_0) = c, \tag{3.4}$$

and the only if part is proven.
(If)
Assume that $(a, b)|c$ and $c = gc_0$. From the fundamental property of polynomial we know that there exist polynomials p and q satisfying

$$ap + bq = g. \tag{3.5}$$

From the above equation, we have

$$c = gc_0 = a(pc_0) + b(qc_0) = ax + by, \tag{3.6}$$

where $x = pc_0$ and $y = qc_0$. Therefor, the equations has the solution.

In the above theorem, we just consider polynomial case. We can extend the result to matrix case. In the matrix case, we have, however, two equations equivalent to (3.1) since matrix multiplication does not commute. The equations are given by

$$XA + YB = C \tag{3.7}$$
$$AX + BY = C, \tag{3.8}$$

where A, B, C, X, Y are all matrices whose elements are all polynomials of d. Concerning to the above equations, we have the following theorem similar to the above theorem.
[Theorem]

Assume that G is the largest right common factor of A and B. (3.7) has a solution if and only if G is also the right common factor of C. (For (3.8), we simply replace right common factor with left common factor.)

From this theorem, in particular if A and B are right (left) coprime, there exist a solution, where right coprimeness (left coprimeness) of A and B means that the largest right (left) common factor is unimodular matrix. We say a matrix M is unimodular if the matrix has inverse in the set of M. In our case, the condition is equivalent to that the determinant of the polynomial matrix is non- zero constant.

The right and left coprimeness can be easily checked by the following conditions. Assume that A and B are polynomial matrices whose sizes are $p \times n$ and $p \times m$. They are left coprime each other if

$$rank[A\ B] = p. \tag{3.9}$$

Assume also A and B are polynomial matrices whose sizes are $n \times p$ and $m \times p$. They are right coprime each other if

$$rank \begin{bmatrix} A \\ B \end{bmatrix} = p. \tag{3.10}$$

As an example, we consider (3.8) where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{3.11}$$
$$B = \begin{bmatrix} 1-d & 2d \\ 1-d & d \end{bmatrix} \tag{3.12}$$
$$C = \begin{bmatrix} 1-d \\ d^2 \end{bmatrix}. \tag{3.13}$$

We can easily check that

$$rank \begin{bmatrix} A \\ B \end{bmatrix} = 2 \tag{3.14}$$

it can be seen that they are right coprime and the equation should have a solution. The one of the solutions is given by

$$X = \begin{bmatrix} 1-d-d^2 \\ d^2 \end{bmatrix} \tag{3.15}$$
$$Y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.16}$$

3.2 d-Step Ahead Predictor

Now we consider a system represented by the following DARMA (Deterministic Auto Regressive Moving Average) model :

$$A(q^{-1})y(k) = B(q^{-1})u(k), \tag{3.17}$$

where

$$
\begin{aligned}
A(q^{-1}) &= I + A_1 q^{-1} + \cdots + A_n q^{-n} && (3.18)\\
B(q^{-1}) &= q^{-d}(B_0 + B_1 q^{-1} + \cdots + A_l q^{-l}) \\
&= q^{-d} B'(q^{-1}) && (3.19)
\end{aligned}
$$

The a d-step ahead prediction can be expressed by

$$
y(k+d) = \alpha(q^{-1})y(k) + \beta(q^{-1})u(k). \tag{3.20}
$$

where

$$
\begin{aligned}
\alpha(q^{-1}) &= G(q^{-1}) && (3.21)\\
\beta(q^{-1}) &= F(q^{-1})B'(q^{-1}), && (3.22)
\end{aligned}
$$

and F and G are the solution of the following Diophantine equation:

$$
F(q^{-1})A(q^{-1}) + G(q^{-1})q^{-d} = I. \tag{3.23}
$$

For the above equation, it can be easily understood that it has a solution if we consider $A = A(q^{-1})$ and $B = q^{-d}$ in the Diophantine equation. Now assume that one of the solutions can be represented as

$$
\begin{aligned}
F(q^{-1}) &= I + F_1 q^{-1} + \cdots + F_{d-1} q^{-d+1} && (3.24)\\
G(q^{-1}) &= G_0 + G_1 q^{-1} + \cdots + G_{n-1} q^{-n+1}. && (3.25)
\end{aligned}
$$

Coefficients of F_i and G_i are easily calculated by the following iteration:

$$
\begin{aligned}
F_0 &= I && (3.26)\\
F_i &= -\sum_{j=0}^{i-1} F_j A_{i-j},\ i = 1, ..., d-1 && (3.27)\\
G_i &= -\sum_{j=0}^{d-1} F_j A_{i+d-j},\ i = 0, ..., n-1, && (3.28)
\end{aligned}
$$

where if A_i $(i > n)$ does appear, we consider it as zero matrix.
[Proof]
As in the above discussion, (3.23) has a solution. Multiplying $F(q^{-1})$ to (3.17) from the left, we have

$$
\begin{aligned}
FAy &= FBu = Fq^{-d}B'u \\
(I - Gq^{-d})y &= Fq^{-d}B'u \\
y &= q^{-d}(Gy + FB'u), && (3.29)
\end{aligned}
$$

and (3.20) is derived. Relationship of the coefficients of F_i and G_i are obtained considering the following equation:

$$
\begin{aligned}
I &= (I + F_1 q^{-1} + \cdots + F_{d-1} q^{-d+1})(I + A_1 q^{-1} + \cdots + A_n q^{-n}) + G_0 q^{-d} + \cdots + G_{n-1} q^{-d-n+1} \\
&= I + (F_1 + A_1)q^{-1} + (F_2 + F_1 A_1 + A_2)q^{-2} + \cdots + (G_0 + A_d + F_1 A_{d-1} + \cdots F_{d-1} A_1)q^{-d} + \cdots && (3.30)
\end{aligned}
$$

Equating the coefficients gives the relationship.

3.3 Adaptive d-Step Ahead Predictor Using Projection Algorithm

Assume that there exists a true parameter vector θ_0 giving the following equation corresponding to a definition of $\phi(k)$.
[Assumption]

$$y(k+d) \;=\; \phi^T(k)\theta_0 \tag{3.31}$$

If a d-step ahead adaptive predictor is constructed by the following equation:

$$\hat{\theta}(k) \;=\; \hat{\theta}(k-1) + \frac{\phi(k-d)}{c+\phi^T(k-d)\phi(k-d)}(y(k)-\phi^T(k-d)\hat{\theta}(k-1)) \tag{3.32}$$

$$\hat{y}(k) \;=\; \phi^T(k-d)\hat{\theta}(k-d), \tag{3.33}$$

then the following properties in the lemma follows.
[Lemma]
Under the following assumptions:

1. $\phi(\cdot)$ is compatible to the orders of the plant. That is, there exists a true parameter vector θ_0 satisfying $y(k+d) = \phi^T(k)\theta_0$.
2. $\{y(k)\}$ and $\{u(k)\}$ are bounded signals,

we have

$$\lim_{N\to\infty}\sum_{k=M}^{M+N}(y(k)-\hat{y}(k))^2 \;<\; \infty \tag{3.34}$$

and $\hat{y}(k) \to y(k)$.
[Proof]
From the properties of the parameter estimation, we have

$$\lim_{N\to\infty}\sum_{k=M}^{N+M}\frac{[\phi^T(k-d)\tilde{\theta}(k-1)]^2}{1+\phi^T(k-d)\phi(k-d)} \;<\; \infty$$

$$\lim_{N\to\infty}\sum_{k=M}^{N+M}\|\hat{\theta}(k)-\hat{\theta}(k-M)\|^2 \;<\; \infty.$$

On the other hand, we have

$$\begin{aligned} e(k) \;&:=\; y(k)-\hat{y}(k) = -\phi^T(k-d)\tilde{\theta}(k-d) \\ &=\; -\phi^T(k-d)\tilde{\theta}(k-1) - \phi^T(k-d)[\hat{\theta}(k-d)-\hat{\theta}(k-1)], \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} e^2(k) \;&=\; (\phi^T(k-d)\tilde{\theta}(k-1))^2 + 2\tilde{\theta}^T(k-1)\phi(k-d)\phi^T(k-d)[[\hat{\theta}(k-d)-\hat{\theta}(k-1)] \\ &+\; (\phi^T(k-d)[\hat{\theta}(k-d)-\hat{\theta}(k-1)])^2 \\ &\le\; 2(\phi^T(k-d)\tilde{\theta}(k-1))^2 + 2(\phi^T(k-d)[\hat{\theta}(k-d)-\hat{\theta}(k-1)])^2 \\ &\le\; 2(\phi^T(k-d)\tilde{\theta}(k-1))^2 + 2\|\phi^T(k-d)\|^2\|\hat{\theta}(k-d)-\hat{\theta}(k-1)]\|^2. \end{aligned} \tag{3.36}$$

Therefor, dividing both sides of the above equation by $(1+\phi^T(k-d)\phi(k-d))$ and summing up with respect to k, we have

$$\sum_{k=1}^{N}\frac{e^2(k)}{1+\phi^T(k-d)\phi(k-d)} \le 2\sum_{k=1}^{N}\frac{[\phi^T(k-d)\tilde{\theta}(k-1)]^2}{1+\phi^T(k-d)\phi(k-d)} + 2\sum_{k=1}^{N}\frac{\|\phi^T(k-d)\|2\|\hat{\theta}(k-d)-\hat{\theta}(k-1)]\|^2}{1+\phi^T(k-d)\phi(k-d)}$$

$$\leq 2\sum_{k=1}^{N} \frac{[\phi^T(k-d)\tilde{\theta}(k-1)]^2}{1+\phi^T(k-d)\phi(k-d)} + 2\sup_{k\geq 1} \|\phi^T(k-d)\|^2 \sum_{k=1}^{N} \|\hat{\theta}(k-d) - \hat{\theta}(k-1)]\|^2 \ < \ \infty. \tag{3.37}$$

From the assumption, $\|\phi(k-d)\|^2$ is bounded, and we have $\sum e^2(k) < \infty$, which shows the lemma.

Chapter 4

Basic Control Method and Adaptive Control

4.1 Basic Control Method

In this chapter we consider following two basic control algorithms:

1. One-Step Ahead Control
2. Model Reference Control

We assume that the objective plant can be expressed by (3.17).

4.1.1 One-Step Ahead Control

One-Step Ahead Control is a method that output of the plant exactly coincides with the desired signal y^* just after d steps where d is a delay. A basic equation for the method is d-step ahead predictor as

$$y(k+d) = \alpha(q^{-1})y(k) \ + \ \beta(q^{-1})u(k). \tag{4.1}$$

The control input is determined by reversing the equation:

$$y^*(k+d) = \alpha(q^{-1})y(k) \ + \ \beta(q^{-1})u(k) \tag{4.2}$$

namely, solving the equation in terms of $u(k)$ as

$$u(k) = \frac{1}{\beta_0}\left\{(\beta(q^{-1}) - \beta_0)u(k) + y^*(k+d) \ - \ \alpha(q^{-1})y(k)\right\}.. \tag{4.3}$$

Note that the control input can be determined by the current and past output, and past input. Concerning about the control, we have the following theorem.
[Theorem]
Using the control given by (4.3), we have

$$y(k) \ = \ y^*(k), \ k \ \geq \ d \tag{4.4}$$

and the input signal is bounded if the following conditions are satisfied.

1. $z^d B(z^{-1})$ does not have zeros outside of a unit circle.
2. Zeros of $B'(z^{-1})/A(z^{-1})$ is within the unit circle.

3. The size of a Jordan block of $B'(z^{-1})$ whose eigenvalue is on the unit circle is one.

[Proof] Omitted.
(Example)
Let consider the following system:

$$\eta(k+1) = \eta(k) \tag{4.5}$$
$$x(k+1) = u(k) \tag{4.6}$$
$$y(k) = x(k) + \eta(k) \tag{4.7}$$

The corresponding $DARMA$ model can be given as

$$y(k) = y(k-1) + u(k-1) - u(k-2), \tag{4.8}$$

and the polynomials A and B are given as

$$A(q^{-1}) = 1 - q^{-1} \tag{4.9}$$
$$B'(q^{-1}) = 1 - q^{-1},\ d = 1. \tag{4.10}$$

For this system the 1-step ahead predictor can be obtained solving the Diophantine equation as

$$F(q^{-1}) = 1,\ G(q^{-1}) = 1. \tag{4.11}$$

Then the control input is expressed as

$$u(k) = y^*(k+1) - y(k) + u(k-1). \tag{4.12}$$

4.1.2 Model Reference Control

In some cases we want to match a input-output relationship (transfer function) to a desired one repressed by a dynamical model. The control method is called Model Reference Control. For the control we assume the following conditions:
[Assumptions]

1. Desired property of the model is represented by
$$E(q^{-1})y^*(k) = gq^{-d'}H(q^{-1})r(k),$$
where $r(k)$, $y^*(k)$ and g are input, output and gain, respectively.
2. $E(z^{-1})$ is stable polynomial.
3. $d' \geq d$.

Before mention about the control method, we require the following lemma.
[Lemma]
The plant considered can be represented as the prediction form as

$$E(q^{-1})y(k+d) = \alpha(q^{-1})y(k) + \beta(q^{-1})u(k), \tag{4.13}$$

where

$$\alpha(q^{-1}) = G(q^{-1}) \tag{4.14}$$
$$\beta(q^{-1}) = F(q^{-1})B'(q^{-1}) \tag{4.15}$$

and F and G are solutions of the following Diophantine equation.

$$E(q^{-1}) = F(q^{-1})A(q^{-1}) + G(q^{-1})q^{-d} \tag{4.16}$$

Keep the above lemma in mind, we have the following theorem.
[Theorem]
Using the following input:

$$\beta(q^{-1})u(k) = E(q^{-1})y*(k+d) - \alpha(q^{-1})y(k) \tag{4.17}$$
$$= E(q^{-1})q^{d-d'}gH(q^{-1})r(k) - \alpha(q^{-1})y(k), \tag{4.18}$$

the output of the plant will approaches to the desired one:

$$y(k) \to y^*(k). \tag{4.19}$$

[Proof]
By the determination of $u(k)$, we have

$$\begin{aligned} E(q^{-1})y(k+d) &= \alpha(q^{-1})y(k) + \beta(q^{-1})u(k) \\ &= E(q^{-1})y^*(k+d). \end{aligned} \tag{4.20}$$

Therefore, we have

$$E(q^{-1})(y(k+d) - y^*(k+d)) = 0, \tag{4.21}$$

and the error will converge to zero because of the stability of the model.

4.2 One-Step-Ahead Adaptive Control

Let consider about adaptive version of the one-step-ahead control taking account of the parameter estimation. Before mention the control scheme, we explain the key technical lemma to show the convergence of the adaptive method.
[Lemma] (The Key Technical Lemma)
Assume that sequences $\{s(k)\}$,$\{\sigma(k)\}$, $\{b_1(k)\}$ and $\{b_2(k)\}$ satisfy the following conditions:

1. $$\lim_{k\to\infty} \frac{s^2(k)}{b_1(k) + b_2(k)\sigma^T(k)\sigma(k)} = 0 \tag{4.22}$$
   where $\{s(k)\}$,$\{b_1(k)\}$ are $\{b_2(k)\}$ are all real sequences and $\{\sigma(k)\}$ is a vector of $p \times 1$.
2. Uniform bounded condition:
   $$0 < b_1(k) < K\infty, \; 0 \leq b_2(k) < K\infty \tag{4.23}$$
   is satisfied for each $k \geq 1$.
3. Linear bounded condition:
   $$\|\sigma(k)\| \leq c_1 + c_2 \max_{0\leq k\leq k} |s(l)| \tag{4.24}$$
   is satisfied where $0 < c_1 < \infty$ and $0 < c_2 < \infty$.

Under the above assumptions, we have

(i) $\lim_{k\to\infty} s(k) = 0$

(ii) $\{\|\sigma(k)\|\}$ is bounded.

[Proof]

(a) Boundness assumption of $\{s(k)\}$ shows the boundness of $\{\|\sigma(k)\|\}$ due to (4.24). Therefore, we have $\lim_{k\to\infty} s(k) = 0$ using (4.23) and (4.22).

(b) Assume that $\{s(k)\}$ is not bounded above. There exists a sub-sequence k_n of k such that

$$\lim_{n\to\infty} s(k_n) = \infty \tag{4.25}$$

and

$$|s(k)| \;\le\; |s(k_n)| \; for \; k \;\le\; k_n. \tag{4.26}$$

($|s(k_n)|$ is non-decreasing sequence with respect to n)

Consider about $|s(k_n)|$ for the sub-sequence $\{k_n\}$, we have

$$\begin{aligned}
\frac{|s(k_n)|}{|[b_1(k_n) \;+\; b_2(k_n)\sigma^T(k_n)\sigma(k_n)]\,|^{1/2}} &\ge \frac{|s(k_n)|}{[K \;+\; K\|\sigma(k_n)\|^2]^{1/2}} \\
&\ge \frac{|s(k_n)|}{K^{1/2} \;+\; K^{1/2}\|\sigma(k_n)\|} \\
&\ge \frac{|s(k_n)|}{K^{1/2} \;+\; K^{1/2}(C_1 \;+\; C_2|s(k_n)|} \\
&\ge \frac{1}{K^{1/2}(1+C_2)/|s(k_n)| \;+\; K^{1/2}C_2}.
\end{aligned} \tag{4.27}$$

Therefore, we have

$$\lim_{k_n\to\infty} \frac{|s(k_n)|}{[|b_1(k_n) \;+\; b_2(k_n)\sigma^T(k_n)\sigma(k_n)|] \;-1/2} \ge \frac{1}{K^{1/2}C_2}. \tag{4.28}$$

The above inequality contradicts the assumption, then $\{s(k)\}$ should be bounded and the lemma is proven.

4.2.1 One-Step-Ahead Adaptive Control

We assume that the plant is represented by (3.17) and the following conditions are satisfied:

1. Delay time d is known.
2. Upper bounds $\bar{n}$ and $\bar{l}$ of n and l are known.
3. The following conditions are satisfied:
   (a) $z^d B(z^{-1})$ has no zeros outside of a unit circle.
   (b) Zeros of $B'(z^{-1})/A(z^{-1})$ are within the unit circle.
   (c) Size of Jordan block of $B'(z^{-1})$ whose eigenvalue are on the unit circle is 1 .

We also assume that the plant can be expressed as

$$A(q^{-1})y(k) = B(q^{-1})u(k) \tag{4.29}$$

where

$$A(q^{-1}) \;=\; I \;+\; A_1q^{-1} \;+\; \cdots \;+\; A_nq^{-n},\; n \;\le\; \bar{n} \tag{4.30}$$

$$B(q^{-1}) \;=\; q^{-d}(B_0 \;+\; B_1q^{-1} \;+\; \cdots \;+\; B_lq^{-l}),\; l \le \bar{l}, \tag{4.31}$$

and the d-step-ahead predictor is given by

$$y(k+d) \;=\; \alpha(q^{-1})y(k) \;+\; \beta(q^{-1})u(k) \;=\; \phi^T(k)\theta_0, \tag{4.32}$$

where

$$\alpha(q^{-1}) \;=\; \alpha_0 \;+\; \cdots \;+\; \alpha_{n-1}q^{-(n-1)} \tag{4.33}$$

$$\beta(q^{-1}) \;=\; \beta_0 \;+\; \cdots \;+\; \beta_{d+l-1}q^{-(d+l-1)},\; \beta_0 \;=\; b_0 \;\ne\; 0, \tag{4.34}$$

and

$$
\begin{aligned}
\phi^T(k) &= [y(k), \cdots, y(k-n+1), u(k), \cdots, u(k-d-l+1)] & (4.35)\\
\theta_0^T &= [\alpha_0, \cdots, \alpha_{n-1}, \beta_0, \cdots, \beta_{d+l-1}]\,. & (4.36)
\end{aligned}
$$

The one-step-ahead adaptive controller is given by the following theorem.
[Theorem]
Parameter estimation and the control input are determined by the following equations:

[Parameter estimation]

$$
\hat{\theta}(k) \;=\; \hat{\theta}(k-1) + a(k)\phi(k-d)[c + \phi^T(k-d)\phi(k-d)]^{-1}(y(k)-\phi^T(k-d)\hat{\theta}(k-1),\ 0 \;<\; a(k) \;<\; 2,\ c \;>\; 0, \quad (4.37)
$$

[Control input]

$$
\phi^T(k)\hat{\theta}(k) \;=\; y^*(k+d), \quad (4.38)
$$

which is

$$
u(k) \;=\; \frac{1}{\hat{\beta}_0(k)}\left[-\hat{\alpha}_0 y(k) - \cdots - \hat{\alpha}_{n-1}y(k-n+1) - \hat{\beta}_1 u(k-1) \cdots - \hat{\beta}_{d+l-1}u(k-d-l+1) + y^*(k+d)\right]. \quad (4.39)
$$

Note that usually we assume $a(k) = 1$, but $a(k)$ is determined by the following equation if $a(k) = 1$ will make $\hat{\beta}_0(k)$ zero:

$$
a(k) \;=\; \gamma,\ \ \epsilon \;<\; \gamma \;<\; 2,\ \gamma \;\neq\; 1 \quad (4.40)
$$

Using the above control method under the assumptions, we have

1. $\{y(k)\}$ and $\{u(k)\}$ are bounded signals.
2. $\lim_{k\to\infty} y(k) \;=\; y^*(k)$。
3. $\lim_{N\to\infty}\sum_{k=d}^{N}(y(k) \;-\; y^*(k))^2 \;<\; \infty$.

[Proof]

Part I From the properties of the parameter estimation, we have

$$
\lim_{k\to\infty} \frac{e(k)}{[c \;+\; \phi^T(k-d)\phi(k-d)]^{1/2}} \;=\; 0 \quad (4.41)
$$

$$
\lim_{k\to\infty} \|\hat{\theta}(k) \;-\; \hat{\theta}(k-d)\| \;=\; 0 \ \ for\ 0 \;<\; d \;<\; \infty \quad (4.42)
$$

where

$$
e(k) \;:=\; y(k) \;-\; \phi^T(k-d)\hat{\theta}(k-1).
$$

Now define $\epsilon(k) \;:=\; y(k) \;-\; y^*(k)$, it can be represented as

$$
\begin{aligned}
\epsilon(k) &= \phi^T(k-d)\theta_0 \;-\; \phi^T(k-d)\hat{\theta}(k-d)\\
&= -\phi^T(k-d)\tilde{\theta}(k-d), & (4.43)
\end{aligned}
$$

where

$$
\tilde{\theta}(k) \;:=\; \hat{\theta}(k) \;-\; \theta_0.
$$

From the above equation, we have

$$
\begin{aligned}
\frac{-\epsilon(k)}{[c\ +\ \phi^T(k-d)\phi(k-d)]^{1/2}} &= \frac{\phi^T(k-d)\tilde{\theta}(k-d)}{[c\ +\ \phi^T(k-d)\phi(k-d)]^{1/2}} \\
&= \frac{\phi^T(k-d)((\hat{\theta}(k-1)-\theta_0)+(\hat{\theta}(k-d)-\theta_0)-(\hat{\theta}(k-1)-\theta_0))}{[c\ +\ \phi^T(k-d)\phi(k-d)]^{1/2}} \\
&= \frac{\phi^T(k-d)\tilde{\theta}(k-1)+\phi^T(k-d)(\tilde{\theta}(k-d)-\tilde{\theta}(k-1))}{[\quad]^{1/2}} \\
&= \frac{-e(k)}{[\quad]^{1/2}}\ +\ \frac{\phi^T(k-d)(\tilde{\theta}(k-d)-\tilde{\theta}(k-1))}{[\quad]^{1/2}},
\end{aligned} \tag{4.44}
$$

and

$$\lim_{k\to\infty}\frac{-\epsilon^2(k)}{[c\ +\ \phi^T(k-d)\phi(k-d)]}\ =\ 0. \tag{4.45}$$

Part II We use the Key Technical Lemma. We consider $s(k) = \epsilon(k)$、$\sigma(k) = \phi(k-d)$、$b_1(k) = c$、$b_2(k) = 1$ in the lemma and the first condition of the Lemma is satisfied and the second condition is also met. Finally, let consider about the third one.

From the assumption of the plant, we have

$$|u(j-d)|\ \le\ m_3\ +\ m_4\max_{1\le l\le k}|y(k)|\ for\ 1\le j\le k,\ 0\ <\ m_3, m_4\ <\ \infty. \tag{4.46}$$

Therefore, taking the property of the elements of $\phi(k-d)$ into account, we have

$$\|\phi(k-d)\|\ \le\ p\{m_3\ +\ max(1,m_4)\max_{1\le l\le k}|y(k)|\}. \tag{4.47}$$

On the other hand, we have

$$|\epsilon(k)|\ \ge\ |y(k)|-|y^*(k)|\ \ge\ |y(k)|\ -\ m_1,\ m_1\ :=\ max|y^*(k)|. \tag{4.48}$$

Therefore, we have

$$
\begin{aligned}
\|\phi(k-d)\| &\le p\{m_3\ +\ max(1,m_4)\max_{1\le l\le k}(|\epsilon(l)|+m_1)\} \\
&= c_1\ +\ c_2 max_{1\le l\le k}|\epsilon(l)|,\ 0\ \le\ c_1\ <\ \infty,\ 0\ <\ c_2\ <\ \infty,
\end{aligned} \tag{4.49}
$$

and the third condition is satisfied. Then from the key technical lemma, it follows that $\epsilon(k)\to 0$ and the boundnesses of $y(k)$ and $u(k)$ from the boundness of $\epsilon(k)$ and (4.46). Also form (4.44), we have

$$\frac{\epsilon^2(k)}{[\quad]}\le 2\left\{\frac{e^2(k)}{[\quad]}\ +\ \frac{\|\phi^T(k-d)(\tilde{\theta}(k-d)-\tilde{\theta}(k-1)\|^2}{[\quad]}\right\} \tag{4.50}$$

and

$$\lim_{N\to\infty}\sum_{k=d}^{N}\epsilon^2(k)\ <\ \infty \tag{4.51}$$

summing up both sides of the above inequality from $k = d$ to N.

Chapter 5

The Kalman Filter and Minimum Variance Control

5.1 The Kalman Filter

5.1.1 Minimum Variance Estimator

Assume that there are two stochastic variables, X and Y, and we know some stochastic properties of them. In that situation we want to estimate X based on the observation of Y. That is, assume $g(Y)$ is a function of the observation Y and we want to find $g()$ such that

$$min_{g(Y)} E\{||g(Y) - X||^2\} \tag{5.1}$$

This $g(Y)$ is called minimum variance estimator of X.

First notice that

$$\begin{aligned} E\{f(X,Y)\} &= \int\int f(x,y)p(x,y)dxdy = \int\int f(x,y)p(x|y)dxp(y)dy \\ &= E\{E\{f(X,Y)|Y\}\}. \end{aligned} \tag{5.2}$$

Hence if we consider $f(X,Y)$ is equal to $||g(Y) - X||^2$, we have

$$\begin{aligned} E\{||g(Y) - X||^2\} &= E\{E\{||g(Y) - X||^2|Y\}\} \\ &= E\{E\{||g(Y) - E\{X|Y\} + E\{X|Y\} - X||^2|Y\}\} \end{aligned} \tag{5.3}$$

Since

$$E\{E\{(g(Y) - E\{X|Y\})(E\{X|Y\} - X)|Y\}\} = E\{(g(Y) - E\{X|Y\})E\{(E\{X|Y\} - X)|Y\}\} \tag{5.4}$$

$$= E\{(g(Y) - E\{X|Y\})(E\{X|Y\} - E\{X|Y\})\} = 0, \tag{5.5}$$

we have

$$E\{||g(Y) - X||^2\} = E\{E\{||g(Y) - E\{X|Y\}||^2\} + E\{||E\{X|Y\} - X||^2\}|Y\}. \tag{5.6}$$

Therefore, the minimum variance estimator is given by the conditional mean as

$$g(Y) = E\{X|Y\}. \tag{5.7}$$

Note that the above result is not affected by the property of the distribution function. In practice, it is very difficult to obtain the conditional mean in general , however, it is easily calculated if two variables are Gaussian as in eq. (5.19). In that equation we consider X_1 is the estimated variable and X_2 is an observed data.

5.1.2 Preliminary

In order to explain how to derive the Kalman filter (Minimum Variance Estimator) we requires some basic matrix calculation and the properties of multi dimensional Gaussian distribution. The properties are shown in this section. First we consider a matrix Σ given by

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix}, \quad \Sigma^T \ = \ \Sigma \tag{5.8}$$

For this Σ we have the following formula:

1. If Σ_{22} is non-singular and $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T$ is also non-singular, the the inverse of Σ is given by

$$\Sigma^{-1} = \begin{bmatrix} X^{-1} & -X^{-1}Y \\ -(X^{-1}Y)^T & Z^{-1} \ + \ Y^T X^{-1} Y \end{bmatrix} \tag{5.9}$$

where

$$\begin{aligned} X &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T \\ Y &= \Sigma_{12}\Sigma_{22}^{-1} \\ Z &= \Sigma_{22} \end{aligned}$$

2.

$$det(\Sigma) \ = \ det(X)det(Z) \tag{5.10}$$

[Proof] Please note that we can factorize Σ into

$$\Sigma \ = \ \begin{bmatrix} I_n & \Sigma_{11}\Sigma_{22}^{-1} \\ 0 & I_m \end{bmatrix} \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T & 0 \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}. \tag{5.11}$$

From this equation, two properties above are easily derived.

As you might know that a n dimensional Gaussian probability density function whose mean is μ and whose covariance matrix is Σ is given by

$$p(X) = \frac{1}{(2\pi)^{n/2}(det(\Sigma))^{1/2}} exp^{-\frac{1}{2}(X-\mu)^T\Sigma^{-1}(X-\mu)}. \tag{5.12}$$

where

$$X = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \xi_n \end{bmatrix}$$

Note that from the property of the probability desity function, we have

$$\int_{-\infty}^{\infty} exp^{-\frac{1}{2}(X-\mu)^T\Sigma^{-1}(X-\mu)} d\xi = (2\pi)^{n/2}(det(\Sigma))^{1/2}. \tag{5.13}$$

Suppose $X_1 \ \in \ R^{n_1}$ and $X_2 \ \in \ R^{n_2}$ and the joint probability density function (p.d.f.) is given by

$$p(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix})(= p(X_1, X_2)) \ = \frac{1}{(\pi)^{(n_1+n_2)/2}(det(\Sigma))^{1/2}} exp^{-\frac{1}{2}\begin{bmatrix} X_1-\mu_1 \\ X_2-\mu_2 \end{bmatrix}^T \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} X_1-\mu_1 \\ X_2-\mu_2 \end{bmatrix}}. \tag{5.14}$$

In order to compute a conditional p.d.f., we modify the exponent as

$$
\begin{aligned}
& \begin{bmatrix} X_1-\mu_1 \\ X_2-\mu_2 \end{bmatrix}^T \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} X_1-\mu_1 \\ X_2-\mu_2 \end{bmatrix} \\
= \; & (X_1-\mu_1)^T X^{-1}(X_1-\mu_1) - (X_1-\mu_1)X^{-1}Y(X_2-\mu_2) - (X_2-\mu_2)^T(X^{-1}Y)^T(X_1-\mu_1) \\
& +(X_2-\mu_2)^T(Z^{-1}+Y^TX^{-1}Y)(X_2-\mu_2) \\
= \; & (X_1-\mu_1-Y(X_2-\mu_2))^T X^{-1}(X_1-\mu_1-Y(X_2-\mu_2)) - (X_2-\mu_2)^T Y^T X^{-1} Y(X_2-\mu_2) \\
& +(X_2-\mu_2)^T(Z^{-1}+Y^TX^{-1}Y)(X_2-\mu_2) \\
= \; & (X_1-\mu_1-Y(X_2-\mu_2))^T X^{-1}(X_1-\mu_1-Y(X_2-\mu_2)) + (X_2-\mu_2)^T Z^{-1}(X_2-\mu_2) \qquad (5.15)
\end{aligned}
$$

where X, Y and Z are defined as before. On the other hand, $p(X_2) = \int p(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix})dx_1$ is computed using the above relation as

$$
p(X_2) = \frac{1}{(2\pi)^{(n_1+n_2)/2}(det(\Sigma))^{1/2}} exp^{-\frac{1}{2}(X_2-\mu_2)^T Z^{-1}(X_2-\mu_2)} \int_{-\infty}^{\infty} exp^{-\frac{1}{2}(X_1-\mu_1-Y(X_2-\mu_2)^T X^{-1}(X_1-\mu_1-Y(X_2-\mu_2)^T)} dx_1. \qquad (5.16)
$$

Using (5.13) and (5.10), we can modify the above eq. as

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{(n_1+n_2)/2}(det(\Sigma))^{1/2}} (2\pi)^{n_1/2}(det(X))^{1/2} exp^{-\frac{1}{2}(X_2-\mu_2)^T Z^{-1}(X_2-\mu_2)} \\
&= \frac{1}{(2\pi)^{n_2/2}(det(Z))^{1/2}} exp^{-\frac{1}{2}(X_2-\mu_2)^T Z^{-1}(X_2-\mu_2)} \qquad (5.17)
\end{aligned}
$$

Therefore, the conditional distribution function $P(X_1, X_2)$ conditioned on $X_2 = x_2$ is given using (5.15) and (5.17) and again (5.10), by

$$
\begin{aligned}
p(\begin{bmatrix} X_1 \\ x_2 \end{bmatrix})(= p(X_1|X_2=x_2)) &= \frac{p(X_1, X_2=x_2)}{p(X_2=x_2)} \\
&= \frac{1}{(\pi)^{n_1/2}(det(X))^{1/2}} exp^{-\frac{1}{2}(X_1-\mu_1-Y(x_2-\mu_2))^T X^{-1}(X_1-\mu_1-Y(x_2-\mu_2))} \qquad (5.18)
\end{aligned}
$$

That is, the conditional mean $E(X_1|X_2)$ and the conditional covariance matrix $cov(X_1, X_1|X_2) = E((X_1 - E(X_1|X_2))(X_1 - E(X_1|X_2))^T|X_2)$ is given as follows:

$$
E(X_1|X_2) = \mu_1 + Y(X_2-\mu_2) \qquad (5.19)
$$

$$
cov(X_1, X_1|X_2) = X \qquad (5.20)
$$

This relationship is the most important formula in the following.

5.1.3 The Kalman Filter

Let consider the following system represented in the state space model:

$$
x(k+1) = \Phi x(k) + \Gamma u(k) + v_1(k) \qquad (5.21)
$$

$$
y(k) = Cx(k) + v_2(k) \qquad (5.22)
$$

where $v_1(k)$ and $v_2(k)$ are system and measurement noise which are both white and Gaussian signals and, we assume the following conditions are satisfied:

$$
E\{x(0)\} = x_0, \;\; cov\{x(0), x(0)\} = \Sigma_0 \qquad (5.23)
$$

$$
E\{\begin{bmatrix} v_1(l) \\ v_2(l) \end{bmatrix} \begin{bmatrix} v_1^T(k) & v_2^T(k) \end{bmatrix}\} = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta(k-l), \;\; Q = Q^T \;\; \geq 0, \;\; R = R^T \;\; > 0 \qquad (5.24)
$$

where $\delta()$ is a Kronecker's delta. We consider an optimal estimator or Kalman filter which estimates the state $x(k+1)$ based on output measurements $\mathcal{Y}(k) = \{y(k), y(k-1), y(k-2), \cdots\}$, namely $\hat{x}(k+1)$ is an estimate of $x(k+1)$ which satisfies

$$min_{\hat{x}(k+1)} E\{||x(k+1) - \hat{x}(k+1)||^2\} = min_{\hat{x}(k+1)} E\{E\{||x(k+1) - \hat{x}(k+1)||^2 | \mathcal{Y}(k)\}\}. \tag{5.25}$$

In order to use the preceding arguments in the previous section, we re-represent (5.21) and (5.22) into a matrix form as

$$\begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} \Phi & I & 0 \\ C & 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ v_1(k) \\ v_2(k) \end{bmatrix} + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k) \tag{5.26}$$

Let assume that the conditional mean of $x(k)$ and the conditional covariance is given as $\hat{x}(k)$ and $\Sigma(k)$. This is true at k = 0 because we assume x_0 and Σ_0 are known. From (5.26) we can calculate the conditional means of $x(k+1)$ and $y(k)$ conditioned on $\mathcal{Y}(k-1)$ as

$$\begin{bmatrix} \hat{x}(k+1) \\ \hat{y}(k) \end{bmatrix} = \begin{bmatrix} \Phi\hat{x}(k) \\ C\hat{x}(k) \end{bmatrix} + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k) \tag{5.27}$$

On the other hand, the conditional covariance is given as follows. Since

$$\begin{aligned} \begin{bmatrix} \tilde{x}(k+1) \\ \tilde{y}(k) \end{bmatrix} &= \begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} - \begin{bmatrix} \hat{x}(k+1) \\ \hat{y}(k) \end{bmatrix} \\ &= \begin{bmatrix} \Phi\tilde{x}(k) + v_1(k) \\ C\tilde{x}(k) + v_2(k) \end{bmatrix} \end{aligned} \tag{5.28}$$

where $\tilde{x}(k) = x(k) - \hat{x}(k)$. Then the conditional covariance is calculated as

$$E\{\begin{bmatrix} \tilde{x}(k) \\ \tilde{y}(k) \end{bmatrix} \begin{bmatrix} \tilde{x}^T(k) & \tilde{y}^T(k) \end{bmatrix} | \mathcal{Y}(k-1)\} = \begin{bmatrix} \Phi\Sigma(k)\Phi^T + Q & \Phi\Sigma(k)C^T + S \\ C\Sigma(k)\Phi^T + S^T & C\Sigma(k)C^T + R \end{bmatrix}. \tag{5.29}$$

Since we have one more observation $y(k)$, we can modify $\hat{x}(k)$ based on $y(k)$. Let consider $x(k+1)$ and $y(k)$ as X_1 and X_2 in (5.18), then we can directory calculate $E\{x(k+1)|y(k)\} = E\{x(k+1)|\{y(k), \mathcal{Y}(k-1)\}\} = E\{x(k+1)|\mathcal{Y}(k)\}$ and its conditional covariance if we consider

$$\begin{aligned} \Sigma &= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} \\ &= \begin{bmatrix} \Phi\Sigma(k)\Phi^T + Q & \Phi\Sigma(k)C^T + S \\ C\Sigma(k)\Phi^T + S^T & C\Sigma(k)C^T + R \end{bmatrix}. \end{aligned} \tag{5.30}$$

That is, using (5.19) and (5.20) we have

$$\hat{x}(k+1) = \Phi\hat{x} + \Gamma u(k) + (\Phi\Sigma(k)C^T + S)(C\Sigma(k)C^T + R)^{-1}(y(k) - C\hat{x}(k)) \tag{5.31}$$

$$\Sigma(k+1) = \Phi\Sigma(k)\Phi^T + Q - (\Phi\Sigma(k)C^T + S)(C\Sigma(k)C^T + R)^{-1}(\Phi\Sigma(k)C^T + S)^T. \tag{5.32}$$

As in the previous section, $\hat{x}(k+1)$ is the conditional mean of the all observations and it is therefore optimal. If we introduce Kalman filter gain $K(k)$ as

$$K(k) = (\Phi\Sigma(k)C^T + S)(C\Sigma(k)C^T + R)^{-1} \tag{5.33}$$

then the Kalman filter is summarized in the following theorem.

[Theorem](The Kalman Filter)

For the system given by (5.21) and (5.22), the optimal filter is given by the following Kalman filter:

$$\hat{x}(k+1) = \Phi\hat{x}(k) + \Gamma u(k) + K(k)(y(k) - C\hat{x}(k))), \ \hat{x}(0) = x_0 \tag{5.34}$$

$$\Sigma(k+1) = \Phi\Sigma(k)\Phi^T + Q - K(k)(C\Sigma(k)C^T + R)K^T(k), \ \Sigma(0) = \Sigma_0 \tag{5.35}$$

$$K(k) = (\Phi\Sigma(k)C^T + S)(C\Sigma(k)C^T + R)^{-1} \tag{5.36}$$

(5.37)

5.1.4 Innovation Process

Let consider the property of the estimation error given by

$$\nu(k) = y(k) - C\hat{x}(k). \tag{5.38}$$

The sequence of $\nu(k)$ is called innovation process. In order to consider the process, we consider the Kalman filter as a system whose input and output are $y(k)$ and $\nu(k)$. That is,

$$\begin{aligned} \hat{x}(k+1) &= (\Phi - K(k)C)\hat{x} \ + \ K(k)y(k) && (5.39) \\ \nu(k) &= -C\hat{x}(k) \ + \ y(k). && (5.40) \end{aligned}$$

Then we can show that the output ν has no correlation, which means that $\nu(k)$ is a white signal even though the input signal $y(k)$ is not. So this system is sometimes called a whiting filter. In order to show the property, we consider $E\{\nu(k)\nu^T(k-i)\}$. Since if we define $\tilde{x}(k) = x(k) - \hat{x}(k)$, the error system is given by

$$\begin{aligned} \tilde{x}(k+1) &= (\Phi - K(k)C)\tilde{x}(k) \ + \ v_1(k) \ - \ K(k)v_2(k) && (5.41) \\ &=: \ \Phi_c(k)\tilde{x}(k) \ + \ v_1(k) \ - \ K(k)v_2(k), && (5.42) \end{aligned}$$

where $\Phi_c(k) = \Phi - K(k)C$, then the state $\tilde{x}(k)$ can be expressed explicitly as

$$\tilde{x}(k) \ = \ H(k-1, k-i)\tilde{x}(k-i) \ + \ \sum_{j=k-i}^{k-1} H(k-1, j+1)(v_1(j) - K(j)v_2(j)) \tag{5.43}$$

where

$$H(k, k-i) = \Phi_c(k)\Phi_c(k-1)\cdots\Phi_c(k-i).$$

Since

$$E\{\nu(k)\nu^T(k-i)\} = CE\{\tilde{x}(k)\tilde{x}^T(k-i)\}C^T + CE\{\tilde{x}(k)v_2^T(k-i)\} + E\{v_2(k)x\tilde{-}i^T(k)\}C^T + E\{v_2(k)v_2^T(k-i)\}, \tag{5.44}$$

the first term in the right-hand side becomes

$$CE\{\tilde{x}(k)\tilde{x}^T(k-i)\}C^T = CH(k-1, k-i)\Sigma(k-i)C^T \tag{5.45}$$

and the second term is

$$CE\{\tilde{x}(k)v_2^T(k-i)\} = -CH(k-1, k-i+1)K(k-i)R. \tag{5.46}$$

The third term is zero since they don't have correlation and the fourth one is trivially $R\delta(i)$. Therefore we have

$$\begin{aligned} E\{\nu(k)\nu^T(k-i)\} &= CH(k-1, k-i+1)(\Phi_c(k-i)\Sigma(k-i)C^T - K(k-i)R) \ + \ R\delta(i) \\ &= CH(k-1, k-i+1)((\Phi - K(k-i)C)\Sigma(k-i)C^T - K(k-i)R) \ + \ R\delta(i) && (5.47) \end{aligned}$$

On the other hand, from the definition of the Kalman filter gain, we have

$$K(k-i)R = \Phi\Sigma(k-i)C^T - K(k-i)C\Sigma(k-i)C^T. \tag{5.48}$$

Substituting the above equation into (5.47), we have

$$E\{\nu(k)\nu^T(k-i)\} = R\delta(i), \tag{5.49}$$

which means that $\nu(k)$ is white.

Up to now, the Kalman filter gain is time varying. Though the gain is time varying but it can be pre-determined, it is suitable for actual implementations that the gain is constant. For the steady state of the Kalman filter, the following fact is known.

[Fact]

Assume (C, A) is detectable and a matrix D is defined by

$$Q \;=\; DD^T. \tag{5.50}$$

If (A, D) is stabilizable and does not have uncontrollable mode on the unit circle, there exist a unique steady state positive definite solution of ARE which stabilize the steady state Kalman filter, where ARE is given by

$$\Sigma \;-\; \Phi\Sigma\Phi^T \;+\; \Phi\Sigma C^T(C\Sigma C^T \;+\; R)^{-1}C\Sigma\Phi^T \;=\; 0 \tag{5.51}$$

5.2 Minimum Variance Control

Let consider a time invariant model of the system based on the Kalman filter represented by

$$\hat{x}(k+1) \;=\; \Phi\hat{x}(k) \;+\; \Gamma u(k) \;+\; K\nu(k) \tag{5.52}$$

$$y(k) \;=\; C\hat{x}(k) \;+\; \nu(k). \tag{5.53}$$

Note that $\nu(k)$ is the innovation process. If we take a $\mathcal{Z}$ transform of the both sides, we have

$$y(z) = C(zI-\Phi)^{-1}\Gamma u(z) \;+\; (C(zI-\Phi)^{-1}K + I)\nu(z). \tag{5.54}$$

From the above equation, we have an ARMAX model as follow:

$$A(q^{-1})y(k) \;=\; B(q^{-1})u(k) \;+\; C(q^{-1})\nu(k) \tag{5.55}$$

where

$$A(q^{-1}) \;=\; 1 \;+\; \alpha_1 q^{-1} \;+\; \cdots \;+\; \alpha_n q^{-n} \tag{5.56}$$

$$B(q^{-1}) \;=\; q^{-d}(\beta_0 \;+\; \beta_1 q^{-1} \;+\; \cdots \;+\; \beta_{n-d}q^{-n+d}) \tag{5.57}$$

$$C(q^{-1}) \;=\; 1 \;+\; \gamma_1 q^{-1} \;+\; \cdots \;+\; \gamma_n q^{-n} \tag{5.58}$$

Note that the coefficients of $C(q^{-1})$ depend on the Kalman filter gain. The minimum variance controller is a controller which minimizes

$$J = E\{||y^*(k+d) - y(k+d)||^2\}. \tag{5.59}$$

First let us consider optimal d-step ahead predictor. Assume that $\alpha(q^{-1})$ and $\beta(q^{-1})$ which satisfy the following equations are given:

$$\alpha(q^{-1}) \;=\; G(q^{-1}) \tag{5.60}$$

$$\beta(q^{-1}) \;=\; F(q^{-1})B'(q^{-1}) \tag{5.61}$$

and $F(q^{-1})$ and $G(q^{-1})$ are the solutions of the following Diophantine equation:

$$A(q^{-1})F(q^{-1}) \;+\; q^{-d}G(q^{-1}) \;=\; C(q^{-1}) \tag{5.62}$$

Note that the highest degree of $F(q^{-1})$ is $d-1$. As usual, multiplying $F(q^{-1})$ both sides of the system and using the Diophantine equation, we have

$$y(k+d) = 1/C(q^{-1})(\alpha(q^{-1})y(k) + \beta(q^{-1})) + F(q^{-1})\nu(k+d). \tag{5.63}$$

On the other hand, $\nu(k)$ is an innovation process and it has no correlation of $\mathcal{Y}(k)$ since the highest degree of $F(q^{-1})$ is $d-1$. Therefore, the optimal prediction of $y(k+d)$ is given by

$$\hat{y}(k+d) = 1/C(q^{-1})(\alpha(q^{-1})y(k) + \beta(q^{-1})u(k)). \tag{5.64}$$

Hence, $\hat{y}(k+d)$ is given by

$$\hat{y}(k+d) = \alpha(q^{-1})y(k) + \beta(q^{-1})u(k) - (C(q^{-1})-1)\hat{y}(k+d). \tag{5.65}$$

Therefore, the optimal control which we can do best is an input which will coincide $\hat{y}(k+d)$ with $y^*(k+d)$, which is

$$u(k) = 1/\beta(q^{-1})(y^*(k+d) - \alpha(q^{-1})y(k) + (C(q^{-1})-1)\hat{y}(k+d)). \tag{5.66}$$

Chapter 6

Model Based Adaptive Control of Robot Manipulators

In this section a basic model basic adaptive control method proposed by J.J. Slotine [6] is first explained in cases where dynamic parameters of the robot manipulators are unknown. This algorithm utilizes properties of the mechanical systems and is an adaptive version of well known "resolved acceleration control". This method does not require acceleration measurements and is very practical for implementations. In this section other two modified methods which are robust for disturbances are also explained. The second one is the one which is robust for system noise [10] and the third one is that which is robust for measurement noise [11]. In order to explain those methods we need some properties of the structure of the robot manipulators, which will be explained in the following sub-section.

We can assume that the dynamic equation of the robot manipulators are expressed, without loss of generality, by :

$$M(q)\ddot{q} \ + \ (C(q,\dot{q}) + D)\dot{q} \ + \ G(q) \ = \ \tau \tag{6.1}$$

where

q	:	Generalized coordinate vector
$M(q)$	:	Inertia matrix
$C(q,\dot{q})$	:	Colioli's and centrifugal force term
D	:	Damping factor
$G(q)$	:	Gravity term
τ	:	Generalized force vector

We should note that the expression of $C(q,\dot{q})$ is not unique. In the following discussion we assume that the state $(q,\dot{q})$ is measurable, and for the simplicity we consider that the term D is zero. For this dynamic model, we have the following properties about each term.[1][11]

[Property 1] $M(q)$ is symmetric.

[Property 2] $M(q)$ is positive defined and there exist constants, λ_m and λ_M, which satisfy the next inequalities if all joints are of revolution.

$$0 \ < \ M_m \ \leq \ \lambda(M(q)) \tag{6.2}$$

$$\lambda(M(q)) \leq M_M \ < \ \infty \tag{6.3}$$

where $\lambda(M)$ is a eigenvalue of a matrix M.
(Later λ is also used as a design parameter but it does not make confusion from the context.)

[Property 3] $C(q,\dot{q})$ is bounded by $\dot{q}$, namely

$$||C(q,\dot{q})|| \ \leq \ C_M||\dot{q}|| \tag{6.4}$$

[Property 4] $C(q, \cdot)$ satisfies the following equalities:

$$\begin{aligned} C(q,x)y &= C(q,y)x & (6.5) \\ C(q,x+\alpha y) &= C(q,x) \ + \ \alpha C(q,y) & (6.6) \end{aligned}$$

[Property 5] $K := \dot{M}(q) - 2C(q,\dot{q})$ is a skew symmetric matrix, namely which satisfies

$$K \ + \ K^T \ = \ 0 \text{ or } x^T K x \ = \ 0 \ \ for \ \forall \ x \tag{6.7}$$

[Property 6] Gravity term, $G(q)$, is bounded for all bounded q.

[Property 7] If we choose a proper parameter vector a of the manipulator, the dynamic equation (6.1) can be represented as a multiplication of a regression matrix Y whose elements can be all calculated using observation signals and the parameter vector a. That is

$$Y(q,\dot{q},\ddot{q})a \ = \ \tau \tag{6.8}$$

In the preceding explanation, we assume the dynamic equation is represented in a joint coordinate space, however, the similar properties are also satisfied in a world coordinate space.

Let consider about the above properties in the case of planar two link manipulator whose dynamic equation is easily derived based on a straight forward calculation, e.g., Lagrangian formulation:

$$\begin{aligned} & \begin{bmatrix} \phi_1+\phi_2+2\phi_3 c_2 & \phi_2+\phi_3 c_2 \\ \phi_2+\phi_3 c_2 & \phi_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \\ & \left(-\phi_3 s_2 \begin{bmatrix} \dot{\theta}_2 & \dot{\theta}_1+\dot{\theta}_2 \\ -\dot{\theta}_1 & 0 \end{bmatrix} + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \right) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\ = & \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \end{aligned} \tag{6.9}$$

where s_2, c_2 stand for $sin(\theta_2)$ and $cos(\theta_2)$ and

$$\begin{aligned} \phi_1 &:= m_1 l_1^2 \ + \ m_2 a_1^2 \ + \ I_1 \\ \phi_2 &:= m_2 l_2^2 \ + \ I_2 \\ \phi_3 &:= m_2 a_1 l_2 \end{aligned}$$

[Property 1] The symmetry is obvious from the dynamic equation (6.9).

[Property 2] The positiveness of the matrix M is shown by calculating a determinant of the matrix as

$$\begin{aligned} & M_{11} \geq \phi_1+\phi_2-2\phi_3 \\ & = m_1 l_1^2 + I_1 + I_2 + m_2(l_2-a_1)^2 \ > \ 0 \\ & detM \geq \phi_1\phi_2 - \phi_3^2 \\ & = m_1 m_2 l_1^2 l_2^2 + m_2 I_1 l_2^2 + (m_1 l_1^2 + m_2 a_1^2 + I_1) I_2 \\ & > \ 0. \end{aligned}$$

Furthermore, the constants M_M, M_m are given as

$$\begin{aligned} M_M &= (\phi_1+\phi_2)\Phi \\ M_m &= \frac{\phi_1\phi_2-\phi_3^2}{\phi_1+2\phi_2+2\phi_3} \end{aligned}$$

where

$$\Phi := max\left\{1, \frac{\phi_2+\phi_3}{\phi_1+\phi_2}\right\}$$

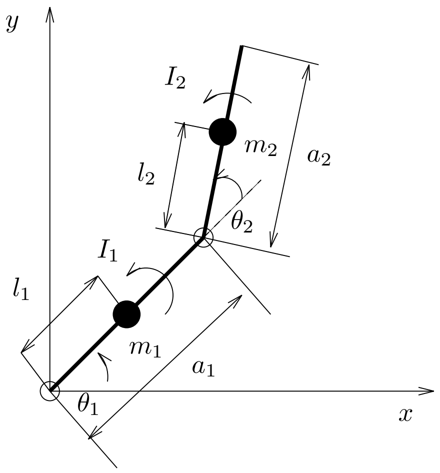

Fig.1 Considered planar two link manipulator

[Property 3] Let consider $x = [x_1, x_2]^T$, we have

$$\begin{aligned}
& x^T C^T(q,\dot{q}) C(q,\dot{q}) x \\
& = \phi_3^2 s_2^2 (x_1^2 \|\dot{q}\|^2 + 2x_1 x_2 \dot{\theta}_2(\dot{\theta_1} + \dot{\theta_2}) + x_2^2(\dot{\theta_1} + \dot{\theta_2})^2) \\
& \le \phi_3^2 (x_1^2 \|\dot{q}\|^2 + 2|x_1 x_2||\dot{\theta}_2(\dot{\theta_1} + \dot{\theta_2})| + x_2^2(\dot{\theta_1} + \dot{\theta_2})^2)
\end{aligned}$$

Using

$$2|a^T b| \ \le \ ||a||^2 \ + \ ||b||^2,$$

we have

$$x^T C^T(q,\dot{q}) C(q,\dot{q}) x \le \phi_3^2 (x_1^2 ||\dot{q}||^2 + ||x||^2 |\dot{\theta}_2(\dot{\theta_1} + \dot{\theta_2})| + x_2^2(\dot{\theta_1} + \dot{\theta_2})^2).$$

Furthermore using this inequality:

$$(a \ + \ b)^2 \ \le \ 2(a^2 \ + \ b^2), \ a, \ b \ \in \ R,$$

we have

$$x^T C^T(q,\dot{q}) C(q,\dot{q}) x \le \phi_3^2 (x_1^2 ||\dot{q}||^2 + ||x||^2 (|\dot{\theta}_2(\dot{\theta_1} + \dot{\theta_2})| + 2x_2^2 ||\dot{q}||^2).$$

The similar alculation shows

$$||C(q,\dot{q})|| \ \le \ 2|\phi_3|||\dot{q}||.$$

[Property 4] Let $x = [x_1, x_2]^T, \ y = [y_1, \ y_2]^T$, we have

$$\begin{aligned}
C(q,x)y &= \phi_3 s_2 \begin{bmatrix} x_2 & x_1 + x_2 \\ -x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
&= \phi_3 s_2 \begin{bmatrix} y_1 x_1 + (y_1 + y_2) x_2 \\ -y_1 x_1 \end{bmatrix} \\
&= \phi_3 s_2 \begin{bmatrix} y_2 & y_1 + y_2 \\ -y_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= C(q,y)x.
\end{aligned}$$

The other equations are also shown similarly.

[Property 5] Calculation of $\dot{M} - 2C$ shows that

$$\begin{aligned} \dot{M} - 2C &= \phi_3 s_2 \begin{bmatrix} -2\dot{\theta}_2 & -\dot{\theta}_2 \\ -\dot{\theta}_2 & 0 \end{bmatrix} \\ &+ \phi_3 s_2 \begin{bmatrix} 2\dot{\theta}_2 & 2\dot{\theta}_1 + 2\dot{\theta}_2 \\ -2\dot{\theta}_1 & 0 \end{bmatrix} \\ &= \phi_3 s_2 \begin{bmatrix} 0 & 2\dot{\theta}_1 + \dot{\theta}_2 \\ -(2\dot{\theta}_1 + \dot{\theta}_2) & 0 \end{bmatrix}. \end{aligned}$$

[Property 6] This term is zero since the manipulator is planar one.

[Property 7] From (6.8), we set $\phi_4 = d_1, \phi_5 = d_2$, and we have

$$\begin{bmatrix} \ddot{\theta}_1 & \ddot{\theta}_1 + \ddot{\theta}_2 & c_2(2\ddot{\theta}_1 + \ddot{\theta}_2) - s_2\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2) & \dot{\theta}_1 & 0 \\ 0 & \ddot{\theta}_1 + \ddot{\theta}_2 & c_2\ddot{\theta}_1 + s_2\dot{\theta}_1^2 & 0 & \dot{\theta}_2 \end{bmatrix}$$
$$\times \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}.$$

Now we have two basic lemmas which play very important roles to show the stabile of the adaptive control algorithms.
(Lemma 1) [12]
Assume that $H(s)$ is a strictly proper stable transfer function whose input is u and whose output is y. If $u \in L_2$, then $y \in L_2 \cap L_\infty$、 $\dot{y} \in L_2$. Furthermore, $y(t) \to 0 \ (t \to \infty)$.

(Lemma 2) [12]
If $y \in L_2$ and $\dot{y} \in L_\infty$, y is uniformly continuous and $y(t) \to 0 \ (t \to \infty)$.
Using these Lemmas, the basic adaptive control algorithm is given in Theorem 1.

(Theorem 1) [6]
Let consider that $q_d, \dot{q}_d, \ddot{q}_d \in L_\infty$ are desired position, velocity and acceleration. We define s (augmented error) and $\dot{q}_r$ (virtual reference) as follows:

$$s := \dot{e} + \Lambda e, \ \ e := q - q_d \tag{6.10}$$
$$\dot{q}_r := \dot{q}_d - \Lambda e. \tag{6.11}$$

If the following control input and parameter update (adaptation) law are used, then q and $\dot{q}$ converge to q_d and $\dot{q}_d$ respectively:

$$\tau = Y_c(q, \dot{q}, \dot{q}_r, \ddot{q}_r)\hat{a} - K_d s \tag{6.12}$$
$$\dot{\hat{a}} = -\Gamma^{-1} Y_c^T(q, \dot{q}, \dot{q}_r, \ddot{q}_r)s \tag{6.13}$$
$$K_d > 0, \ \ \Gamma > 0$$

where
$$Y_c(q, \dot{q}, \dot{q}_r, \ddot{q}_r) := \hat{D}\ddot{q}_r + \hat{C}(q, \dot{q})\dot{q}_r + \hat{G}(q), \tag{6.14}$$

and ˆ means that the corresponding term is calculated using estimated parameters. Please note that in this control law we do not require acceleration signal.
(Note)
In the algorithm we used the word 'parameter update'. Please note, however, that it is not assumed the estimated

parameter converge to true ones, and the adaptive control law can be considered as a kind of direct adaptive control method in a sense.
(Proof)
Let $\tilde{a}$ be a difference between the true parameter and the estimated one as

$$\tilde{a} \;:=\; \hat{a} \;-\; a, \tag{6.15}$$

and consider the following Lyapunov function candidate for s and $\tilde{a}$: (In the following the arguments are omitted for the simplicity.)

$$V_1 := \frac{1}{2} s^T M(q) s \;+\; \frac{1}{2}\tilde{a}^T \Gamma \tilde{a} \tag{6.16}$$

Taking a derivative of V_1 gives us

$$\dot{V_1} = s^T M(\ddot{q} - \ddot{q}_r) + \frac{1}{2} s^T \dot{M} s + \tilde{a}^T \Gamma \dot{\tilde{a}}.$$

Using the relationship of (6.1) and noting that a is a constant vector and $\dot{\tilde{a}} = \dot{\hat{a}}$, we have

$$= s^T(\tau - C\dot{q} - G) - s^T M \ddot{q}_r + \frac{1}{2} s^T \dot{M} s + \tilde{a}^T \Gamma \dot{\hat{a}}.$$

Using the property 5 and the definition of $\dot{q}_r$, we have

$$= s^T(\tau - M\ddot{q}_r - C\dot{q}_r - G) \;+\; \tilde{a}^T \Gamma \dot{\hat{a}}.$$

Substituting the control law and the parameter updation law into the above equation, we have

$$= -s^T K_d s \;\leq\; 0. \tag{6.17}$$

From this inequality, it is obvious that $V_1(t)$ is monotonically decreasing function and s and $\hat{a}$ are bounded signals. Furthermore, integrating the above inequality, we have

$$||s||^2 \;\leq\; \frac{1}{\lambda_d} V_1(0), \;\;\; \lambda_d := min \lambda(K_d),$$

which shows $s \in L_2$. Let now consider that the definition of s is a first order state space differential equation whose input is s and whose state is e. Then the transfer function from u to e is a strictly proper stable transfer function and we can use Lemma 1. The lemma shows that $e \rightarrow 0$ and e and $\dot{e} \;\in\; L_\infty$ and $\dot{e} \in L_2$. Furthermore, from the conditions, it is shown that $q, q_r, \dot{q}, \dot{q}_r$ are all bounded, and using

$$||M\ddot{q}|| = ||\tau - C\dot{q} - G||$$

and the roundedness of $\hat{a}$ and the property 2, we can show $\ddot{q}$ is bounded signal. From these properties we have $\ddot{e} \in L_\infty$, and it follows that $\dot{e} \rightarrow 0$ from Lemma 2,. This completes the proof.

[]

The adaptive control method given by Theorem 1 takes advantage of properties of the dynamics of the mechanical systems and is very sophisticated one in theory. In practice it has some difficulties for the actual implementations. One is that since the parameter updation law is just a integration law, the convergence of the parameters is not good in general. For this problem, J.J. Slotine has proposed a similar adaptive control method but it incorporates the least square parameter estimation law. The other problem is that in order to derive the control law, any disturbance, e.g., system noise or observation noise, is not assumed in the system. In the real situations, however, there always exist the disturbances. In the linear systems, bounded noise signal does not cause instability problem, however, in the non-linear system as the adaptive control system, they sometimes cause instability phenomena. In such a sense, the adaptive control method is not robust.

In general we use the following three methods to make adaptive control algorithms robust for the disturbances.

1. Introducing dead zone for parameter updation law.
2. Normalization of the signals used for parameter updation law.
3. Introducing a feedback term in the parameter updation law.(σ modification)

In the parameter updation law, error signals (augmented error signals) are used and the error signal become small when the parameters converges to the true ones without the disturbances. They does not becomes small when there exist miss-match between the model and the actual plant, then the persistent error will make the estimated parameters diverge. In that case we can not expect that the error signal becomes smaller than a certain value. For that case, the error signal should be considered zero even though the signal is not zero, and the parameter is not updated. The zone of the certain value of the error signal is called dead zone. The second method is a method to avoid parameter divergence introducing a normalization factor and a constant number as a denominator in the updating law as in the least square method. The third method is a method to avoid parameter divergence introducing a feedback term in the parameter updation law, which is given in the following theorem. (Theorem 2) [10]
We assume that a bounded system noise d exists in the dynamic equation as

$$M(q)\ddot{q} \ + \ (C(q,\dot{q}) + D)\dot{q} \ + \ G(q) \ + \ d(t) \ = \ \tau. \tag{6.18}$$

where $d(.)$ is a function of only time and it satisfies

$$||d||_\infty \ = \ \bar{d}. \tag{6.19}$$

We also assume that a norm of the parameter vector is bounded above by

$$||a|| \ \leq \ \bar{a}, \tag{6.20}$$

and $\bar{a}$ is known. If the control law and the parameter updation law are modified as

$$\begin{aligned} \tau &= Y_c(q,\dot{q},\dot{q}_r,\ddot{q}_r)\hat{a} - K_d s, K_d > 0 & (6.21)\\ \dot{\hat{a}} &= -\Gamma^{-1}(Y_c^T(q,\dot{q},\dot{q}_r,\ddot{q}_r)s \ + \ \sigma\hat{a}), \\ & \quad \Gamma > 0, \ \sigma \ > \ 0, & (6.22) \end{aligned}$$

where $\Lambda^T K_d$ is a symmetric and positive define and σ_0 is a certain positive scaler and σ is determined as

$$\sigma \ = \ \begin{cases} 0 & \text{if} \quad ||\hat{a}|| \ < \ \bar{a} \\ \sigma_0 \left(\frac{||\hat{a}||}{\bar{a}} - 1\right) & \text{if} \quad \bar{a} \ < \ ||\hat{a}|| \ \leq \ 2\bar{a} \\ \sigma_0 & \text{if} \quad 2\bar{a} \ < \ ||\hat{a}|| \end{cases}, \tag{6.23}$$

then $(e,\dot{e})$ does converge into the region expressed by

$$D \ := \ \left\{(e,\dot{e}) \mid ||e||^2 + \lambda_{min}||\dot{e}||^2 \ \leq \ \frac{\bar{d}^2}{\beta^2}(1+c^2), \right\} \tag{6.24}$$

where

$$\begin{aligned} \lambda_{min} &:= min\lambda(\Lambda), \ \lambda_{max} := max\lambda(\Lambda), \\ c &:= \frac{\lambda_{max}}{\lambda_{min}}, \ \beta := min\lambda(K_d). \end{aligned}$$

(Note)
This theorem shows that if the disturbance is large, then the region will become naturally big. On the other hand, the values of Λ, and K_d, which are gain and design parameters, are big, the region will become small. When $c = 1$ and the manipulator is of one degree of freedom, the region of the convergence is directed as in Fig. 2.

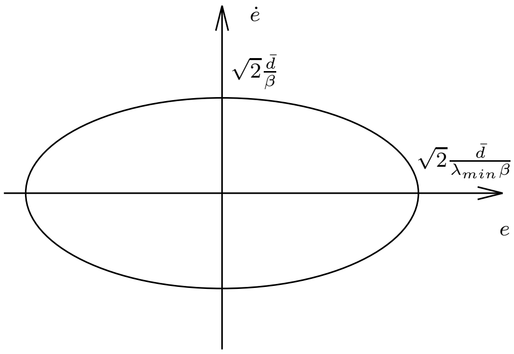

Fig.2 Region of the convergence.

(Proof)
Modifying the Lyapunov candidate in Theorem 1 as

$$V_2 := \frac{1}{2}s^T M(q)s \; + \; e^T\Lambda^T K_d e \; + \; \frac{1}{2}\tilde{a}^T\Gamma\tilde{a}. \tag{6.25}$$

Taking the derivative of V_2 and following the previous derivations in the proof of Theorem 1, we have

$$\dot{V}_2 \;=\; -\dot{e}^T K_d \dot{e} \;-\; e^T\Lambda^T K_d \Lambda e \;-\; \sigma\tilde{a}^T\hat{a} \;+\; s^T d. \tag{6.26}$$

Due to the determination of σ, we have $\sigma\tilde{a}^T\hat{a} \geq 0$ since

$$\begin{aligned}&\sigma(\hat{a}-a)^T\hat{a} = \sigma(||\hat{a}||^2 - a^T\hat{a})\\ &\geq \sigma(||\hat{a}||^2 - ||a||||\hat{a}||)\\ &\geq \sigma||\hat{a}||(||\hat{a}|| - \bar{a}) \;\geq\; 0.\end{aligned} \tag{6.27}$$

Therefore, we have

$$\begin{aligned}&\dot{V}_2 \leq -\dot{e}K_d\dot{e} - e^T\Lambda^T K_d\Lambda e + s^T d\\ &\leq -\beta||\dot{e}||^2 - \beta\lambda_{min}^2||e|| + ||\dot{e}||||d|| + \lambda_{max}||e||||d||\\ &\leq -\beta||\dot{e}||^2 - \beta\lambda_{min}^2||e|| + ||\dot{e}||\bar{d} + \lambda_{max}||e||\bar{d}\end{aligned}$$

In order to cancel out $||\dot{e}||\bar{d}, ||e||\bar{d}$ from the above equation, completing a square we have

$$\begin{aligned}&\dot{V}_2 \leq -\frac{\beta}{2}||\dot{e}||^2 - \frac{\beta}{2}(||\dot{e}|| - \frac{\bar{d}}{\beta})^2 \;+\; \frac{\bar{d}^2}{2\beta}\\ &-\frac{\beta\lambda_{min}^2}{2}||e||^2 \;-\; \frac{\beta}{2}(\lambda_{min}||e|| - \frac{\lambda_{max}\bar{d}}{\lambda_{min}\beta})^2 + \frac{\beta\lambda_{max}^2\bar{d}^2}{2\lambda_{min}^2}\\ &\leq \frac{\beta}{2}\left(-||\dot{e}||^2 \;-\; \lambda_{min}^2||e||^2 \;+\; \frac{\bar{d}^2}{\beta^2}(1 + \frac{\lambda_{max}^2}{\lambda_{min}^2})\right).\end{aligned} \tag{6.28}$$

Since in the region of $\|\dot{e}\|^2 + \lambda_{min}^2||e||^2 \;>\; \frac{\bar{d}^2}{\beta^2}(1+c^2)$, $\dot{V}_2 \;<\; 0$, then $(e,\dot{e}) \;\rightarrow\; D$.

[]

In theorem 2, we just considered the system noise, in practice there exists observation noise. In particular, if we obtain velocity signal taking a difference of the position signal, the quantization error sometimes leads to drift of the parameter signal. This phenomena is crucial when the desired signal is constant and the position error is small.

Let consider this phenomena in the basic adaptive control method. Let assume that e is zero and $\dot{e}$ is n being the observation noise, and the parameter updation law becomes as

$$\dot{\hat{a}} = -\Gamma^{-1} Y_c^T(q, n, 0, -\Lambda(n))(n). \tag{6.29}$$

Please notice that there exists a term consisting of the square of n, and this term becomes the source of the drift. For this problem, the following Theorem give a solution.
(Theorem 3) [11]
Assume that the control input is determined by

$$\begin{aligned} \tau &= Y_c(q, \dot{q} - \lambda e, \dot{q}_d, \ddot{q}_d)\hat{a} - K_d\dot{e} - K_p e && (6.30) \\ \dot{\hat{a}} &= \Gamma^{-1} Y_c(q, \dot{q} - \lambda e, \dot{q}_d, \ddot{q}_d)s && (6.31) \end{aligned}$$

where

$$K_d = K_d^T > 0, \ K_p = K_p^T > 0.$$

Assume also that $\Lambda = \lambda I$ and

$$\lambda := \frac{\lambda_0}{1 + ||e||} \tag{6.32}$$

and

$$0 < \lambda_0 < min\left\{\frac{K_{dm}}{(3M_M + 2C_M)}, \frac{4K_{pm}}{(K_{dM} + K_{dm})}\right\}. \tag{6.33}$$

where

$$\begin{aligned} K_{dm} &:= min\lambda(K_d), \ K_{dM} := max\lambda(K_d), \\ K_{pm} &:= min\lambda(K_p), \ K_{pM} := max\lambda(K_p), \end{aligned}$$

M_m, M_M, C_M are constants in (Property 2) and (Property 3). Then e and $\dot{e}$ does converges to zero in large and all internal signals are bounded.
(Note)
In this algorithm, since we need M_m, M_M, C_m, the algorithm is not a pure adaptive controller in a sense. The resultant control algorithm is just a PD controller with gravity compensation. From this structure, please notice that there does not exist square term of n but that of the desired value.
(Proof)
Substituting the control input into (6.1), we have

$$\begin{aligned} & M\ddot{e} + C(q, \dot{q})\dot{e} + \lambda C(q, e)\dot{q}_d + K_d\dot{e} - K_p e \\ = \ & Y_c(q, \dot{q} - \lambda e, \dot{q}_d, \dot{q}d_d)\tilde{a}. \end{aligned} \tag{6.34}$$

Let now consider the following Lyapunov function candidate:

$$V_3 = \frac{1}{2}s^T M(q)s + \frac{1}{2}e^T K_p e + \frac{1}{2}\tilde{a}^T \Gamma \tilde{a}. \tag{6.35}$$

Taking the derivative of it, we have

$$\begin{aligned} & \dot{V}_3(e, \dot{e}, \tilde{a}) \\ & = s^T \Big[\lambda M(q)\dot{e} + \dot{\lambda} M(q)e + \lambda C(q, \dot{q})e \\ & -\lambda C(q, e)q_d - K_d\dot{e} - K_p e] + \dot{e}^T K_p e. \end{aligned} \tag{6.36}$$

Introducing the following a new variable s_1 defined by

$$s_1 \equiv \dot{e} + \frac{\lambda}{2}e, \tag{6.37}$$

the above equation can be re-expressed using the variable as

$$\dot{V}_3(e,\dot{e},\tilde{a}) = s_1^T\left[\lambda M(q)\dot{e} + \dot{\lambda} M(q)e + \lambda C(q,\dot{q})e, -\lambda C(q,e)q_d - K_d\dot{e} - K_p e\right] + \dot{e}^T K_p e. \tag{6.38}$$

Now consider the second and third terms, we have the following inequalities using the determination of λ and the property of $C(q,.)$:

$$\dot{\lambda} s^T M(q)e \leq 2\lambda_0 M_M(\|s_1\|^2 + \|\frac{\lambda}{2}e\|^2) \tag{6.39}$$

$$\lambda s^T C(q,\dot{e})e \leq 2\lambda_0 C_M(\|s_1\|^2 + \|\frac{\lambda}{2}e\|^2). \tag{6.40}$$

From these inequalities, we have

$$\dot{V}_3(e,\dot{e}) \leq -\kappa_1\|s_1\|^2 - \kappa_2\|\frac{\lambda}{2}e\|^2. \tag{6.41}$$

where

$$\kappa_1 \equiv K_{d,m} - 3\lambda_0 M_M - 2\lambda_0 C_M \tag{6.42}$$

$$\kappa_2 \equiv \frac{4}{\lambda_0}K_{p,m} - K_{d,M} - 2\lambda_0 M_M - 2\lambda_0 C_M. \tag{6.43}$$

From the condition of λ_0, we have $\dot{V}_3 \leq 0$ and V_3 is a monotonically decreasing function. As in Theorem 1, we can show that $e, s_1 \in L_\infty^n$, and $\lambda \in L_\infty^n$ and $e \in L_2^n$, $\dot{e} \in L_\infty$. Therefore, from Lemma 1 we have $e \to 0$ $(t \to \infty)$. We can also show the convergence of $\dot{e}$ as in a similar way in Theorem 1.

$\square$

In actual implementations the computation of $Y_c()$ is the most difficult part. The computation is, however, relatively easily executed using spatial notation. Since this discussion is beyond the scope of this manuscript, please refer to the reference. [7]

References

1. M.W.Spong and M.Vidyasagar, Robot Dynamics & Control, John Wiley & Sons, (1989)

2. J.J.E.Slotine and W.Li, Applied Nonlinear Control, Prentice Hall, (1991)

3. J.J.Craig, Adaptive Control of Robot Manipulators, Addison-Wesley

4. H.Seraji, "Adaptive Independent Joint Control of Manipulators: Theory and Experiment, IEEE Int. Conf. on R. & A., 854/861, (1988)

5. M.W.Walker, "Adaptive Control of Manipulators Containing Closed Kinematic Loops", IEEE Trans. on R. & A., Vol. 6, No.1, 10/19, (1990)

6. J.J.Slotine and W.Li, "On the Adaptive Control of Robot Manipulators", Int. Joutnal on Robotics Research, Vol. 6, 49/59, (1987)

7. G.Niemeyer and J.J.Slotine, "Computational Algorithms for Adaptive Computation", IEEE CDC, 566/571, (1989)

8. M.Ghodoussi and Y.Nakamura, "Principal Base Parameters of Open and Closed Kinematic Chains, IEEE Int. Conf. on R. & A., 84/89

9. S.Sastry and M.Bodson, Adaptive Control: Stability, Convergence, and Robustness, Prentice-Hall, (1988)

10. M.Zribi and S.Ahmad, "Robust Adaptive Control of Multiple Robots in Cooperative Motion Using σ Modification", IEEE Int. Conf. on R. & A., 2160/2165, (1991)

11. H.Berghuis, R.Ortega and H.Nijmeijer, "A Robust Adaptive Controller for Robot Manipulators", Int. Conf. on R. & A., 1876/1881, (1992)

12. C.A.Desoer and M.Vidyasagar, Feedback System: Input-Output Properties, Academic Press, (1975)